Math 313 - Analysis I **Spring 2009**

HOMEWORK #3
DUE NOON, JANUARY 30, 2009

(1) Let $a_n = \frac{2n}{n+1}$, for $n = 1, 2, 3, \ldots$. Using the formal definition of limit, prove that $\lim_{n\to\infty} a_n = 2$.

(2) Define the sequence b_n as follows:

$$b_n = \begin{cases} \frac{n-1}{n}, & \text{if } n \text{ is odd, } n \geq 1, \text{ and} \\ \\ \frac{n+1}{n}, & \text{if } n \text{ is even, } n \geq 2. \end{cases}$$

Again using the formal definition of limit, prove that $\lim_{n\to\infty} b_n = 1$.

(3) Prove that if $\lim_{n\to\infty} c_n = L$ for some $L \in \mathbb{R}$ then $\lim_{n\to\infty} c_n^3 = L^3$. Conversely, prove that if $\lim_{n\to\infty} c_n^3 = L^3$, then $\lim_{n\to\infty} c_n = L$.

On the other hand, give an example of a sequence (c_n) and a real number L where $\lim_{n\to\infty} c_n^2 = L^2$ but where it is not true that $\lim_{n\to\infty} c_n = L$.

(4) Suppose (d_n) is a sequence such that $\lim_{n\to\infty} d_n = L$, where L is a real number. Define the sequence (e_n) by $e_n = d_{n^2}$. Prove that $\lim_{n\to\infty} e_n = L$.

(5) Buniakowski's Inequality is the statement that for any finite interval $[a, b] \subseteq \mathbb{R}$ and any two "well-behaved" functions $f(x)$ and $g(x)$ we have the following inequality:

$$\left| \int_a^b f(x)g(x)dx \right|^2 \leq \int_a^b (f(x))^2 \, dx \int_a^b (g(x))^2 \, dx.$$

Show that Buniakowski's Inequality implies the Cauchy-Schwarz inequality. (Hint: divide the interval $[a, b]$ into n equal parts, and choose f and g to have constant values on each part. You may assume that these functions are "well-behaved" so Buniakowski's Inequality does indeed hold.)