Math 313 - Analysis I **Spring 2009**

WORKED SOLUTIONS TO MIDTERM #1

(1) Suppose that a and b are real numbers so that $b > 3$.

(a) Prove that $\left|\frac{a}{3} - \frac{3}{b}\right| < \frac{|ab-9|}{9}$.

(b) Now suppose that $3 < b < 9$, and prove that $\left|\frac{a}{3} - \frac{3}{b}\right| < |a-3| + \frac{|b-3|}{3}$.

Solution:

NOTE: I missed the possibility that $ab = 9$, in which case both $\frac{a}{3} - \frac{3}{b}$ and $ab - 9$ are zero, so the inequality fails. Sorry about this!

(a) $\left|\frac{a}{3} - \frac{3}{b}\right| = \left|\frac{ab-9}{3b}\right| = \frac{|ab-9|}{3b} < \frac{|ab-9|}{9}$, since $b > 3$ (so long as $ab-9 \neq 0$).

(b) By the first part, we have $\left|\frac{a}{3} - \frac{3}{b}\right| < \frac{|ab-9|}{9}$, so it suffices to prove that $\frac{|ab-9|}{9} < |a-3| + \frac{b-3}{3}$. Well,

$$\begin{aligned}
\frac{|ab-9|}{9} &= \frac{|ab-3b+3b-9|}{9} \\
&\leq \frac{|ab-3b|}{9} + \frac{|3b-9|}{9} \\
&= \frac{b}{9}\,|a-3| + \frac{|b-3|}{3} \\
&< |a-3| + \frac{|b-3|}{3},
\end{aligned}$$

since $b < 9$.

(2) Let A be a nonempty finite subset of $\mathbb{R}$. Prove, by induction on the size of A, that A is bounded below.

[Note: If you like, you may assume that given $a, b \in \mathbb{R}$ there is a number $\min\{a,b\} \in \mathbb{R}$ so that $\min\{a,b\} \leq a$ and $\min\{a,b\} \leq b$. (And it is equal to either a or to b). However, **do not** assume that a finite set has a minimum, since this is basically what you are being asked to prove...]

Solution:

The proof is by induction on the size of the set A.

Base case: A has one element, a_1 say. Then a_1 is clearly a lower bound for A.

The inductive hypothesis is: Suppose that $k \geq 1$ and that all subsets of $\mathbb{R}$ of size k have a lower bound.

For the inductive step, let $A = \{a_1, \ldots, a_{k+1}\}$ be a subset of $\mathbb{R}$ of size $k+1$. We must show that A is bounded below.

Define $B = \{a_1, \dots, a_k\}$. By induction, B has a lower bound, which we'll call d.

Let $b = \min\{d, a_{k+1}\}$. We claim that b is a lower bound for A. Well, let $x \in A$. Then either

(i) $x \in B$, so $b \leq d \leq x$ (since d is a lower bound for B); or

(ii) $x = a_{k+1}$, so $b \leq a_{k+1}$.

Therefore, in any case we have $b \leq x$, so b is a lower bound for A as required.

Therefore, we have proved that all finite nonempty subsets of $\mathbb{R}$ are bounded below, by induction.

(3) Let (a_n) be a sequence of real numbers. Suppose that $\{a_n \mid n \in \mathbb{N}\}$ is bounded below and $B = \inf\{a_n \mid n \in \mathbb{N}\}$. Suppose further that $\lim_{n\to\infty} a_n = L$. Prove that $B \leq L$.

Give an example where $B \neq L$. (You do not need to prove that your example works, but you should say what B and L are.)

Solution:

Suppose, in order to obtain a contradiction, that $B > L$.

Since $\lim\limits_{n\to\infty} a_n = L$ we know that for all $\epsilon > 0$ there is an $N \in \mathbb{N}$ so that for all $n \geq N$ we have

$$|a_n - L| < \epsilon.$$

Let $\epsilon = B - L$, which is greater than zero since we assumed $B > L$. Let $N \in \mathbb{N}$ be as in the above statement. Then for $n \geq N$ we have

$$|a_n - L| < B - L,$$

which implies that $a_n < L + (B - L) = B$. This contradicts the fact that $B = \inf\{a_n \mid n \in \mathbb{N}\}$ (in particular B is a lower bound for $\{a_n\}$). Therefore we must have $B \leq L$, as required.

For an example, let a_n be defined by $a_1 = 0$, $a_n = 1$, for $n > 1$. Then $B = 0$ and $L = 1$.

(4) Prove that $\sum\limits_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = 1$.

Solution:

Note that

$$\begin{aligned} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} &= \frac{\sqrt{n+1}}{\sqrt{n}\sqrt{n+1}} - \frac{\sqrt{n}}{\sqrt{n}\sqrt{n+1}} \\ &= \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \end{aligned}$$

Therefore, the partial sum S_n is

$$\begin{aligned}
S_n &= \sum_{i=1}^{n} \frac{\sqrt{i+1}-\sqrt{i}}{\sqrt{i^2+i}} \\
&= \sum_{i=1}^{n} \left(\frac{1}{\sqrt{i}} - \frac{1}{\sqrt{i+1}} \right) \\
&= (1-\frac{1}{\sqrt{2}}) + (\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}}) + \ldots + (\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}) \\
&= 1 - \frac{1}{\sqrt{n+1}}.
\end{aligned}$$

Clearly,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(1 - \frac{1}{\sqrt{n+1}} \right) = 1,$$

so we have proved that

$$\sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n^2+n}} = 1,$$

as required.