SOLUTIONS TO HW #7

Chapter 7

22. Suppose that G is a group with more than one element and G has no proper, nontrivial subgroups. Prove that $|G|$ is prime. (Do not assume at the outset that G is finite.)

Solution.
Let $|G| \geq 2$ (possibly $|G| = \infty$). If G has no proper nontrivial subgroups, then G and $\langle e \rangle$ are the only subgroups. Let $a \in G$ be a nonidentity element. Then the subgroup generated by a cannot be $\langle e \rangle$, so $\langle a \rangle = G$, hence G is cyclic. If $|G| = \infty$, then $G \cong \mathbb{Z}$. But $\mathbb{Z}$ has nontrivial proper subgroups. Thus $|G| < \infty$. Suppose $|G| = n$. Since G is cyclic (that is a CRUCIAL point), for EVERY divisor d of n, $\exists$ a subgroup H of order d. If $d \neq 1$ or $d \neq n$, then H is a proper, nontrivial subgroup of G. Therefore, the only divisors of n are n and 1, hence n is prime.

24. Let G be a group of order 25. Prove that G is cyclic or $g^5 = e$ for all g in G.

Solution.
If G is cyclic, then we're done. So assume that G is not cyclic. Let $g \in G$. If $g = e$, then clearly $g^5 = e$. So suppose $g \neq e$. Then $|g|$ divides 25, *i.e.*, $|g| = 1$, 5, or 25. But $|g| \neq 1$ since we assumed $g \neq e$, and $|g| \neq 25$ since otherwise, G would be cyclic. So $|g| = 5$, *i.e.*, $g^5 = e$.

30. Prove that every subgroup of D_n of odd order is cyclic.

Solution.
Suppose H is a subgroup of D_n, and $|H| = m$, where m is odd. Then $m \mid 2n$. But m odd means $m \nmid 2$, so $m \mid n$. The elements of D_n are rotations and reflections. If $q \in D_n$ is a reflection, then $q^2 = e$, *i.e.*, $|q| = 2$. But $2 \nmid m$ since m is odd, so q cannot belong to H. This means that the only possible elements of H are rotations. Consider the subset K of D_n consisting of all rotations (including the identity). This is a subgroup of D_n of order n. Notice that K is in fact cyclic (it is generated by $R_{2\pi/n}$). Since H is a subgroup of D_n that can only contain rotations, H is a subgroup of K, a cyclic subgroup of D_n. Hence H is cyclic.

40. Let G be the group of rotations of a plane about a point P in the plane. Thinking of G as a group of permutations of the plane, describe the orbit of a point Q in the plane.

Solution.
If P is fixed and G is the group of rotations of a plane about P, then Q traces a circle around P of radius $|PQ|$. The reason for this is that rotation by *any* angle about P preserves the symmetry of the plane, so we obtain every possible point in the plane that lies at a distance of $|PQ|$ from P, in other words, a circle centered at P with radius $|PQ|$.

42. We will go in order from left to right on the first row and left to right on the second row for this problem. "North," "South," "East," "West," "Northeast (NE)," "Northwest (NW)," "North by Northwest (NNW)," etc. will indicate where points in the orbit are. Your solution should have one picture for each square, ideally indicating which group element sends the original point to the indicated point. Let d_1 denote the diagonal running from NW to SE and d_2 the diagonal running from NE to SW.

(i) The orbit is {East, North, West, South}. The stabilizer is $\{R_0, Q_H\}$ since Q_H keeps the original point "East" in its position. (Notice that the orbit has 4 elements, the stabilizer 2, and $4 \cdot 2 = 8$, the order of D_4; this is consistent with the Orbit-Stabilizer Theorem.)

(ii) Orbit: {NE, NW, SW, SE}. Stabilizer: $\{R_0, Q_{d_2}\}$.

(iii) Orbit: {E, N, W, S}. Stabilizer: $\{R_0, Q_H\}$.

(iv) Orbit: {ENE, NNE, NNW, WNW, WSW, SSW, SSE, ESE}. Stabilizer: $\{R_0\}$. (Notice |orbit|=8, |stabilizer|=1, and again, $8 \cdot 1 = 8$.)

(v) Same as (iv) except the point is inside the small triangular region instead of on the boundary.

(vi) Same as (v), but skewed a little.

44. Use the Orbit-Stabilizer Theorem and choose a "convenient" point from which to do your calculations, typically either a vertex or the "center" of a polygonal face.

a. Regular tetrahedron: Choose a vertex, say the "top" one. Then |stabilizer|=3, since you may rotate the tetrahedron $0, 2\pi/3$, or $4\pi/3$ radians about the axis through the top vertex and keep it where it is. Notice that a symmetry would have to take a vertex to another vertex or the center of a face to the center of another face. So, |orbit|=4 since there are 4 vertices. Hence, the order of the rotation group of the tetrahedron is $3 \cdot 4 = 12$.

b. Regular octahedron: Choose, say, the top vertex. Then |stabilizer|=4, since you may rotate $\pi/2$ radians at a time about the axis through the top vertex and preserve symmetry. Then there are 6 vertices to which you may send the top vertex (including itself) via a rotational symmetry, so |orbit|=6. Thus, the group of rotations has order $4 \cdot 6 = 24$.

c. Regular dodecahedron: Choose the "center" of a pentagonal face. You can rotate the dodecahedron about the axis through this point by $2\pi n/5$, $n = 0, 1, 2, 3, 4$ for a total of 5 elements in the stabilizer. You can take this face to any other face by a rotational symmetry, and there are 12 faces, so |orbit|=12. Hence, the rotation group of the dodecahedron has order $5 \cdot 12 = 60$.

d. Regular icosahedron: Choose a vertex, rotate by $2\pi n/5$, $n = 0, 1, 2, 3, 4$ to obtain |stabilizer|=5. (Notice that the similarity to the previous calculation stems from the fact that the dodecahedron and icosahedron are dual solids.) You can take this vertex to any other vertex by a rotational symmetry, and here there are 12 vertices, so |orbit|=12. Thus, the order of the rotation group is $5 \cdot 12 = 60$.