Math 330 - Abstract Algebra I **Spring 2009**

MIDTERM #2 (PRACTICE)
1PM, APRIL 3, 2009

(1) Let S_7 be the symmetric group of degree 7, consisting of all permutations of the set $A = \{1, 2, 3, 4, 5, 6, 7\}$.

Let $f : A \to A$ be defined by the following table:

$$\begin{array}{cccccccc} x & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ f(x) & 2 & 4 & 5 & 1 & 7 & 6 & 3 \end{array}$$

- (a) Write f as a product of disjoint cycles.
- (b) What is the order of f?
- (c) Write f as a product of 2-cycles. Is f even or odd?
- (d) Calculate f^{76}, written as a product of disjoint cycles.

Solution:

(a): We have $f(1) = 2$, $f(2) = 4$, $f(4) = 1$, $f(3) = 5$, $f(5) = 7$, $f(7) = 3$ and $f(6) = 6$. Therefore, we can write f as

$$f = (1\ 2\ 4)(3\ 5\ 7)$$

(we could also write it as $(124)(357)(6)$, but that's not the way we normally write it, we can also write the cycles in a different order, if we like...).

(b): The order of a permutation, written as a product of disjoint cycles, is the least common multiple of the lengths of the cycles. In this case we have a product of two 3-cycles, so the order is 3.

(c): We can write:

$$f = (57)(37)(24)(14),$$

so f is even.

(d): Since the order of f is 3, we have $f^{75} = (f^3)^{25} = e^{25} = e$. Therefore, $f^{76} = f$ and

$$f^{76} = (1\ 2\ 4)(3\ 5\ 7).$$

(2) Suppose that G is an abelian group, and $H \leq G$ is a subgroup.

- (a) Prove that H is normal in G
- (b) Prove that G/H is abelian.

(c) Give an example of a group G with an abelian normal subgroup H so that G is not abelian, but G/H is abelian.

Solutions:

(a): We know that H is abelian if and only if for each $g \in G$ we have $gH = Hg$. Well,

$$\begin{aligned} gH &= \{gh \mid h \in H\} \\ &= \{hg \mid h \in H\}, \text{ since } G \text{ is abelian,} \\ &= Hg. \end{aligned}$$

Therefore H is normal.

(b): Suppose that $g_1, g_2 \in G$ then

$$\begin{aligned} (g_1H)(g_2H) &= (g_1g_2)H \text{ by definition of the product in } G/H \\ &= (g_2g_1)H\text{since } G \text{ is abelian,} \\ &= (g_2H)(g_1H). \end{aligned}$$

Therefore, G/H is abelian.

(c): Let $G = S_3$, the symmetric group on 3 letters, and let

$$H = \langle(123)\rangle = \{e, (123), (132)\}.$$

Then H is normal in G and is abelian. Also, G/H has order 2, and we know all groups of order 2 are abelian. But G itself is nonabelian, so this is an example as required.

(3) (a) Suppose that G_1 is a group and $g, h \in G_1$. Prove that if

$$(gh)^2 = g^2h^2$$

then

$$gh = hg.$$

(b) Suppose that G_2 is a group so that $g^2 = e$ for all $g \in G_2$. Prove that G is abelian.

(c) Let G_3 be a group and suppose that all nonidentity elements of G_3 have order 2. Suppose that G_3 has size at least 3. Prove that G_3 has a subgroup of order 4.

(d) Let G_4 be a group of order 22. Prove that G_4 has an element of order 11.

Solutions:

(a):

$$\begin{aligned} gh &= (g^{-1}g)gh(hh^{-1}) \\ &= g^{-1}(g^2h^2)h^{-1} \\ &= g^{-1}(gh)^2h^{-1} \\ &= g^{-1}(ghgh)h^{-1} \\ &= hg \end{aligned}$$

(b): Let g and h be elements of G_2. Then we know that $(gh)^2 = 1$, $g^2 = 1$ and $h^2 = 1$. Therefore, $(gh)^2 = g^2h^2$. By Part (a), we have $gh = hg$. Since g and h were arbitrary, G_2 must be abelian.

(c): Let g, h be two elements of G_3 so that neither of them are e and $g \neq h$. Consider the set $H = \{e, g, h, gh\}$.

We can multiply:

$$gg = e, hh = e, gh = gh, (gh)h = g, (gh)g = h,$$

etc. and easily show that H is a subgroup of G_3. (This uses the fact that G_3 is abelian so $(gh)g = g^2h = h$, for example) But maybe H does not have size exactly 4. We know that H has size at least 3. Suppose that $|H| = 3$. Then the order of any element of H must be 1 or 3. However, it has elements of order 2. Hence $|H| \neq 3$ and we must have $|H| = 4$, so H is the subgroup of G_3 of order 4, as required.

d: The possible orders of elements of G_4 are 1, 2, 11 and 22. There is only one element of order 1. If there is an element g in G_4 so that $|g| = 22$. Then g^2 has order 11.

So suppose that there is no element of order 22 and no element of order 11. Then all nontrivial elements of G_4 have order 2. By Part (c), there is a subgroup of G_4 of order 4. But 4 does not divide 22, so this contradicts Lagrange's Theorem. Therefore, there is an element of G_4 of order 11, as required.