Math 330 - Abstract Algebra I **Spring 2009**

MIDTERM #2
SOLUTIONS

(1) Let S_9 be the symmetric group of degree 9, consisting of all permutations of the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Let $\alpha : A \to A$ be defined by the following table:

x	1	2	3	4	5	6	7	8	9
$\alpha(x)$	3	5	4	6	2	9	8	7	1

(a) Write α as a product of disjoint cycles.

(b) What is the order of α?

(c) Calculate α^{102}, as a product of disjoint cycles.

Solutions:

(a)

$$\alpha = (13469)(25)(78).$$

(b) The order of a product of disjoint cycles is the least common multiple of the lengths of the cycles. So the order of α is 10, since there is a 5-cycle and a pair of 2-cycles in the decomposition of α.

(c) Since $\alpha^{10} = e$, we have $\alpha^{102} = \alpha^2$, which is

$$(14936).$$

(2) Consider D_4 as a subgroup of permutations of the corners of a square (so $D_4 \leq S_4$).

(a) Find an odd permutation in D_4.

(b) What is the subgroup of D_4 of even permutations?

[Hint: Remember from the Spring Break Practice Problems that if the subset of even permutations in D_4 is not all of D_4 then it consists of exactly half the elements of D_4. You may assume this result without proof.]

Solutions

(a) Consider the four vertices to be labelled $1, 2, 3, 4$, with 1 opposite 3 and 4 opposite 2. The reflection through the line through 1 and 3 is given by the permutation (24). This is a single 2-cycle, so is odd.

(b) Since there is an odd permutation, the set of even permutations in D_4 must be half the elements of D_4. Therefore, since $|D_4| = 8$, the subgroup of even permutations must have size 4. As permutations, here are four such elements:

$$\{e, (12)(34), (14)(23), (13)(24)\}$$

Geometrically, these are:

(a) The identity map;
(b) Reflection through the line crossing through the sides joining 1 to 2 and joining 3 to 4;
(c) Reflection through the line crossing through the sides joining 1 to 4 and joining 2 to 3; and
(d) Rotation by π.

(3) Consider the group $\mathbb{Z}/15\mathbb{Z}$, and let $g = 6 + 15\mathbb{Z} \in G$.

(a) What is the order of g in G?
(b) List the elements of $\langle g \rangle$, the cyclic subgroup of G generated by g.
(c) What is the factor group $G/\langle g \rangle$? (Give an explicit group and an explicit isomorphism from G to it).

Soutions:

(a) We could work it out by using the theory of cyclic groups, but let's just do it from scratch:

$$\begin{aligned} g &\neq e \\ g^2 = 12 + 15\mathbb{Z} &\neq e \\ g^3 = 18 + 15\mathbb{Z} &\neq e \\ g^4 = 24 + 15\mathbb{Z} &\neq e \\ g^5 = 30 + 15\mathbb{Z} &= e. \end{aligned}$$

So the order of g is 5. (This is also 15 divided by the greatest common divisor of 6 and 15.)

(b) We've just listed the elements above, but here they are again (written in a more sensible way and order):

$$\{15\mathbb{Z}, 3 + 15\mathbb{Z}, 6 + 15\mathbb{Z}, 9 + 15\mathbb{Z}, 12 + 15\mathbb{Z}\}$$

(c) The order of $G/\langle g \rangle$ is $\frac{15}{5} = 3$. There is only one group of order 3, up to isomorphism, which is the cyclic group of order 3.

Let's write the cyclic group of order 3 as integers mod 3, and list its elements as $\{0 + 3\mathbb{Z}, 1 + 3\mathbb{Z}, 2 + 3\mathbb{Z}\}$.

The elements of $G/\langle g \rangle$ are cosets of $\langle g \rangle$ in $\mathbb{Z}/15\mathbb{Z}$. These cosets can be written as:

$$\{(0 + 15\mathbb{Z}) + \langle g \rangle, (1 + 15\mathbb{Z}) + \langle g \rangle, (2 + 15\mathbb{Z}) + \langle g \rangle\}$$

The isomorphism between $G/\langle g \rangle$ and $\mathbb{Z}/15\mathbb{Z}$ is given by

$$(i + 15\mathbb{Z}) + \langle g \rangle \mapsto i + 3\mathbb{Z}$$