The Logarithm Function

The logarithm function is the reverse of the exponential function:

Suppose that $b > 0$, and $b \neq 1$. If x is a positive number, the logarithm of x to the base b, denoted $\log_b(x)$, is the number y so that

$$b^y = x.$$

So,

$$\log_b(x) = y$$

if and only if

$$b^y = x.$$

COMMENT: On your calculater, the

'log' button does $\log_{10}$, and the

'ln' button does $\log_e$.

Rules for logarithms

Suppose that $b > 0$ and $b \neq 1$. Then

$$\log_b(1) = 0$$

and

$$\log_b(b) = 1.$$

Rules for logarithms, contd.

For any positive numbers u and v we have:

Equality: $\log_b(u) = \log_b(v)$ if and only if $u = v$.

Product: $\log_b(uv) = \log_b(u) + \log_b(v)$.

Power: For any real number r we have $\log_b(u^r) = r \log_b(u)$.

Quotient: $\log_b\left(\frac{u}{v}\right) = \log_b(u) - \log_b(v)$.

Inversions: $\log_b(b^u) = u$ and $b^{\log_b(v)} = v$.

We now want to consider the logarithm as a function:

$$y = \log_b(x).$$

• The graph of $y = \log_b(x)$ is obtained from the graph of $y = b^x$ by reversing x and y. What this means is that you **reflect** the graph of $y = b^x$ across the line $y = x$.

PICTURES of graphs...

Properties of the logarithm function:

Suppose that $b > 0$ and $b \neq 1$. Then the function $y = \log_b(x)$ satisfies:

- It is defined and continuous for all $x > 0$.

[But it is undefined for $x = 0$ and $x < 0$...]

- The y-axis is a vertical asymptote.

- The x-intercept is $(1, 0)$ and there is no y-intercept.

- If $b > 1$ then

$$\lim_{x \to 0^+} \log_b(x) = -\infty$$

and

$$\lim_{x \to \infty} \log_b(x) = \infty.$$

On the other hand, if $0 < b < 1$ we have

$$\lim_{x \to 0^+} \log_b(x) = \infty,$$

and

$$\lim_{x \to \infty} \log_b(x) = -\infty.$$

- If $b > 1$ then $y = \log_b(x)$ is increasing for all x.

If $0 < b < 1$ then $y = \log_b(x)$ is decreasing for all x.

The Natural Logarithm

Suppose that $x > 0$. The **natural logarithm of** x if

$$\ln(x) = \log_e(x).$$

In other words

$$y = \ln(x)$$

means that

$$e^y = x.$$

We have

$$e^{\ln(x)} = x$$

(for $x > 0$) and

$$\ln(e^y) = y,$$

(and all y).

We can convert between logarithms of different bases:

Suppose $b \neq 1$ and

$$
\begin{aligned}
c &= \log_b(a). \text{ Then,} \\
b^c &= a \text{ , by definition. Now, taking the natural logarithm of both} \\
\ln(b^c) &= \ln(a)\text{, so} \\
c\ln(b) &= \ln(a)\text{, and we get} \\
c &= \frac{\ln(a)}{\ln(b)}
\end{aligned}
$$

In summary, we get:

Conversion Formula for Logarithms

If a and b are positive numbers (and $b \neq 1$) then

$$\log_b(a) = \frac{\ln(a)}{\ln(b)}.$$

Application: Doubling time

If we have $Q(t) = Q_0 e^{kt}$ (like continuous compound interest) then the size of Q doubles in time d, where

$$d = \frac{\ln(2)}{k}.$$

Example: Suppose that the interest rate is 5%, and interest is compounded continuously. Then your investment grows as

$$Q(t) = Q_0 e^{0.05t}$$

where Q_0 is the initial investment, and your investment will double in

$$d = \frac{2}{0.05} \approx 13.86$$

years.