Times other than doubling time

Suppose that our quantity $Q(t)$ satisfies the law:

$$Q(t) = Q_0 e^{kt}$$

for $t \geq 0$.

Suppose we want to know how long it takes to increase by a factor of P. (Or decrease if k is negative. Let's do k positive first.) Then to find this time, we set $Q(t_P) = PQ_0$. This gives:

$$PQ_0 = Q_0 e^{kt_P}$$

so $P = e^{kt_P}$, which means $t_P = \frac{\ln P}{k}$.

This will only work if k is positive, and $P \geq 1$. (Otherwise we get negative time.)

Suppose we have $P < 1$ and $k < 0$ (for example 'halving-time' or 'half-life'). Then let's write $Q(t) = Q_0 e^{-rt}$ (where $r = -k$ we get (for $P = \frac{1}{2}$):

$$t_{\frac{1}{2}} = \frac{\ln\left(\frac{1}{2}\right)}{k} = \frac{\ln(2)}{r}.$$

For other kinds of decaying time (other than halving) we put in $\ln(P)$ instead of $\ln(\frac{1}{2})$.

Here's a short list of places where exponential growth or decay can arise:

- Population
- Compound interest
- GDP growth (In the U.S., the historical average is about 2% per year
- Ponzi schemes
- Amway
- Computer processing power (Moore's Law).

Graphs of functions involving exponentials and logarithms

Example: Sketch the graph of

$$y = x^2 - 3x - 2\ln(x) + 5$$

The Normal Distribution

The **normal distribution** satisfies the function:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2}$$

(when it has average 0 and standard deviation 1). For the function to determine the 'curve for grades', it will be something like:

$$f(x) = \frac{1}{\sqrt{30\pi}} e^{\frac{-(x-65)^2}{30}}$$

[Average 65 standard deviation 15.]

What can we say about the graph of this function (say the first one)?

The Learning Curve

Suppose that A, B and k are all positive numbers. Then the graph of the function

$$Q(t) = B - Ae^{-kt}$$

is called the learning curve.

It grows from $B - A$ at time $t = 0$ to a horizontal asymptote of $y = B$ (so it approaches but never quite reaches B).

PICTURE

The learning curve is everywhere increasing and always concave downward.

The Logistic Curve

Suppose that A, B and k are positive numbers. The **logistic curve** is the graph of

$$Q(t) = \frac{B}{1 + Ae^{-Bkt}}$$

This is used for many things – for example modelling population when there is a maximum population (called **'carrying capacity'**).

The Logistic Curve grows from $\frac{B}{1+A}$ at time $t = 0$ and has a horizontal asymptote at $y = B$. PICTURE

It is everywhere increasing, and has a point of inflection at

$t = \frac{\ln(A)}{Bk}$.

Example:

Suppose that a new species of deer is introduced into Wisconsin, and that after t years, the number of deer (in thousands) is

$$D(t) = \frac{100}{1 + 99e^{-0.1t}}$$

(a) How many deer were originally introduced? How many were there after 2 years? After 20 years?

(b) When does the rate of change of deer begin to decline?

(c) What is the maximum number of these deer that Wisconsin can carry?