Integration by Substitution

One thing that can sometimes make integrals easier (or possible to calculate at all!) is changing variables. Suppose that we have to calculate:

$$\int f(x)dx$$

We can define a new variable $u = u(x)$ for some function $u(x)$ (which is supposed to make things easier). Then we have

$$\frac{du}{dx} = u'(x),$$

and we write

$$du = u'(x)dx.$$

We use this equation to substitute into the original equation, and hopefully can now solve it. This is best illustrated with examples.

Examples

Solve the following integrals.

$$\int (4x-3)^5 dx$$

$$\int \sqrt{2x+1}\,dx$$

$$\int \frac{x^2+1}{x-1}dx$$

$$\int (x+1)e^{x^2+2x}dx$$

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isn't, and you really just have to expand this out into some huge ugly polynomial.

Here are some steps to try with integration by substitution to find $\int f(x)dx$:

(1) Guess a substitution $u = u(x)$ which is supposed to make life easier.

(2) Rewrite $f(x)$ in terms of u and $du = u'(x)dx$.

(3) Now we should have

$$\int f(x)dx = \int g(u)du,$$

so try to find an antiderivative $G(u)$ for $g(u)$.

(4) Substitute $u(x)$ for u in $G(u)$ to find that $G(u(x))$ is an antiderivative of $f(x)$. So we have

$$\int f(x)dx = G(u(x)) + C.$$

More examples:

$$\int \frac{\ln(x)^5}{3x} dx$$

$$\int x\sqrt{4x+1}\,dx$$

$$\int \frac{e^x}{\sqrt{e^x - 5}} dx$$

$$\int (e^x + 2x - 1)(e^x + x^2 - x + 3)^2 dx.$$