Rules for definite integrals:

- **Constant multiple rule:** For a constant k,

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx$$

- **Sum rule:**

$$\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

- **Difference rule:**

$$\int_a^b f(x) - g(x)dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

- $$\int_a^a f(x)dx = 0$$

- $$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

- **Subdivision rule:**

$$\int_a^b f(x)dx = \int_a^c f(x)dx = \int_c^b f(x)dx$$

Example

Suppose that $f(x)$ and $g(x)$ are continuous functions so that

$$\int_1^3 f(x)dx = 3, \int_2^3 f(x)dx = 1, \int_1^3 f(x) + g(x)dx = 5.$$

What is

$$\int_1^3 g(x)dx?$$

What is

$$\int_1^2 f(x)dx?$$

Substitution for a definite integral:

Integration by substitution works when our function $f(x)$ can be expressed as

$$f(x) = g(u(x)).u'(x)$$

for some functions $g(y)$ and $u(x)$. Then the Chain Rule says that if $G(u)$ is an antiderivative of $g(u)$ then $G(u(x))$ is an antiderivative of $f(x)$.

In other words:

$$\int f(x)dx = \int g(u(x))u'(x)dx = \int g(u)du,$$

(where we then substitute back in $u = u(x)$ to get a function for $x\ldots$)

Phew!

It is easier for definite integrals.

Suppose we have functions f, g and u so that

$$f(x) = g(u(x)).u'(x),$$

as before. Then:

$$\int_a^b f(x)dx = \int_a^b g(u(x)).u'(x)dx = \int_{u(a)}^{u(b)} g(u)du$$

In other words, after making the substitution $u = u(x)$, we can determine the new limits for the integral, and ignore the variable x entirely.

[Alternatively, as the book describes, we can use integration by substitution to find the antiderivative $F(x) = G(u(x))$ directly, and then evaluate $F(b) - F(a)$...]

Let's see these two approaches through examples:

Examples

Evaluate the following definite integrals:

$$\int_1^2 2x(x^2-2)^3 dx$$

$$\int_0^1 2xe^{x^2} dx$$

$$\int_1^3 \frac{2x-1}{x^2-x+3} dx$$

Net Change

Suppose that $Q(x)$ is a function so that $Q'(x)$ is continuous on the interval $a \le x \le b$. The **net change** in $Q(x)$ between $x = a$ and $x = b$ is

$$Q(b) - Q(a) = \int_a^b Q''(x) dx.$$

Example

A particle moves so that at time t it has position $p(t)$ on the x-axis. Suppose that the velocity function is

$$v(t) = p'(t) = 3t^2 - 2t + 5.$$

What is the net change for the particle between time $t = 1$ and $t = 3$?