(Continuous) Compound Interest

Consider an investment of P dollars which is invested at an interest rate of r, expressed as a decimal (so 5% is expressed as 0.05). And suppose that the interest is paid k times per year. Then each period the interest rate is $\frac{r}{k}$.

Then, after 1 period, a payment of $P \times \frac{r}{k}$ is paid, and the total amount is

$$P_1 = P + \left(P \times \frac{r}{k}\right) = P \times \left(1 + \frac{r}{k}\right).$$

After the second period, a payment $P_1 \times \frac{r}{k}$ is made, and the total is then

$$P_2 = P_1 \times \left(1 + \frac{r}{k}\right) = P\left(1 + \frac{r}{k}\right)^2.$$

We proceed like this, and in general, we get the following:

Compound interest formula Suppose P dollars are invested at an interest rate r and interest is compounded k times per year. If $B(t)$ is the value of the investment after t years (called **future value**) then

$$B(t) = P\left(1 + \frac{r}{k}\right)^{kt}.$$

If interest is compounded continuously then

$$B(t) = Pe^{rt}$$

The formula for continuous compounding is explained as follows:

Let $n = \frac{k}{r}$. Then as we get more and more periods in each year, the value of n gets bigger and bigger.

We'll consider a limit of the first formula as the number of periods per year goes to infinity. (Think: Interest paid every quarter, then month, then day, then hour, then second, then millisecond, etc.)

Now, the first formula says that with k periods per year we have

$$\begin{aligned} B(t) &= P\left(1+\frac{r}{k}\right)^{kt} \\ &= P\left(1+\frac{1}{n}\right)^{nrt} \\ &= P\left[\left(1+\frac{1}{n}\right)^{n}\right]^{rt}. \end{aligned}$$

(Note that $\frac{r}{k} = \frac{1}{n}$ and $kt = nrt$.)

Now we let $n \to \infty$ and we get (for continuous compounding)

$$B(t) = \lim_{n\to\infty} P\left[\left(1+\frac{1}{n}\right)^{n}\right]^{rt} = Pe^{rt}$$

which is the formula we wanted.

Example:

Suppose that we have \$1000 to invest for five years, and we have two choices:

- Interest rate of 5% (0.05) compounded quarterly; or
- Interest rate of 4.5% (0.045) compounded continuously.

Which one should we do?

[Answer worked out on next slide...]

Answer:

For the first one, we apply the first formula as follows:

$P = 1000$, $t = 5$, $k = 4$, $r = 0.05$. So after five years we will have

$$B(5) = P\left(1 + \frac{r}{k}\right)^{kt}$$

Answer:

For the first one, we apply the first formula as follows:

$P = 1000$, $t = 5$, $k = 4$, $r = 0.05$. So after five years we will have

$$\begin{aligned} B(5) &= P\left(1+\frac{r}{k}\right)^{kt} \\ &= 1000\left(1+\frac{0.05}{4}\right)^{4\times 5} \\ &= 1000 \times 1.0125^{20} \\ &= 1000 \times 1.282037... \\ &= 1282.04 \end{aligned}$$

So under the first scheme we would have \$1282.04.

On the other hand, for the second scheme we have:

$P = 1000, t = 5, r = 0.05$ and then we get

$$
\begin{aligned}
B(5) &= Pe^{rt} \\
&= 1000 \times e^{0.05 \times 5} \\
&= 1000 \times e^{0.25} \\
&= 1000 \times 1284025... \\
&= 1284.03
\end{aligned}
$$

So we would have \$1284.03 and this is the better option, by almost \$2.

Another thing we can do with these formulae is work out how much to invest now if we know how much we want (and the interest rate and the number of years):

This is known as the **present value** since it can also be used (with inflation taken into account in the interest...) to work out how much B dollars in future money is worth now...

Present Value:

The present value of B dollars in T years invested at the annual rate r compounded k times per year is

$$P = B\left(1 + \frac{r}{k}\right)^{-kT}.$$

If interest is compounded continuously then we get

$$P = Be^{-rt}.$$

Examples...