Partial Derivatives

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Suppose that we have a function $f(x, y)$ of two variables x and y. There are lots of ways to measure the infinitesimal rate of change of f.

We're going to concentrate on the **partial derivatives.**

Definition: The partial derivative of f with respect to x at a point (a, b) is:

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h},$$

supposing that this limit exists.

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[**IMPORTANT: The two appearances of 'x' in $f_x(x, y)$ serve two different purposes...**]

Similarly, we have the partial derivative of f with respect to y:

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h},$$

and as a function:

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

Alternative notation:

$$\frac{\partial f}{\partial x} = f_x$$

$$\frac{\partial f}{\partial y} = f_y$$

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Well, just like for single-variable functions, there are lots of rules.

In fact, the best thing to do to compute f_x is to pretend that y is just some constant and differentiate as if it was just a function of x...

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Let's do some examples.

Examples:

For the following functions, find the partial derivatives with respect to x and y:

$$\begin{aligned} f(x,y) &= x^2 + xy + y^2 \\ f(x,y) &= 3x^2y - 5xy^3 + 7x - 3 \\ f(x,y) &= e^{xy} \\ f(x,y) &= xe^y \\ f(x,y) &= x\ln(x^2 + y) \end{aligned}$$

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$$f_{xx} = (f_x)_x = \frac{\partial^2 f}{\partial x \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$
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$$f_{yy} = (f_y)_y = \frac{\partial^2 f}{\partial y \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

Aaagghh!

OK, there are four second partial derivatives:

$$f_{xx}$$

$$f_{xy}$$

$$f_{yx}$$

$$f_{yy}$$

OK, there are four second partial derivatives:

$$f_{xx}$$

$$f_{xy}$$

$$f_{yx}$$

$$f_{yy}$$

The only difficult thing is to remember which order to do the derivatives in f_{xy} and f_{yx}.

In fact, for all reasonable functions that we'll see:

$$f_{xy} = f_{yx},$$

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Let's do some examples of second partial derivatives.

Example:

Compute all of the first and second partial derivatives of the following functions: [And check that $f_{xy} = f_{yx}$.

$$
\begin{aligned}
f(x, y) &= x^2y - yx + 3y \\
f(x, y) &= xe^{xy} \\
f(x, y) &= \frac{x + y}{x - y}
\end{aligned}
$$