Review

Review

Plan for this week:

Review

Plan for this week:

• Today, we'll run briefly over the material we've covered this semester, and then go over the basic features of the various topics.

Review

Plan for this week:

- Today, we'll run briefly over the material we've covered this semester, and then go over the basic features of the various topics.

- This will run over into Wednesday. During this time, I'll give some examples. During the remaining time on Wednesday and Friday, we'll work these examples. Of course, if you have things that you particularly want to go over, we can do this later in the week.

Review

Plan for this week:

- Today, we'll run briefly over the material we've covered this semester, and then go over the basic features of the various topics.

- This will run over into Wednesday. During this time, I'll give some examples. During the remaining time on Wednesday and Friday, we'll work these examples. Of course, if you have things that you particularly want to go over, we can do this later in the week.

In the review I'm going to give today, it probably won't help much if you aren't already pretty familiar with what we've done this semester, because there's a ton of material so we'll go quickly.

Functions

We spent some time talking about the definition of functions, but for us they are given by a formula, or sometime formulae. The **domain** of a function consists of all the points where it is defined. [The x for which $f(x)$ exists.]

Functions

We spent some time talking about the definition of functions, but for us they are given by a formula, or sometime formulae. The **domain** of a function consists of all the points where it is defined. [The x for which $f(x)$ exists.]

In practice for us, this means the x when we are not forced to:

1. Take the square root of a negative number;

2. Divide by 0;

3. Take $\ln(a)$ for $a \leq 0$.

Example:

What is the domain of the following function?

$$f(x) = \begin{cases} \frac{1}{x^2-3x-10} & \text{if } x \leq -3 \\ \ln(3x+4) & \text{if } -3 < x < 0 \\ \sqrt{x^2+2x} & \text{if } 0 \leq x \leq 2 \end{cases}$$

Limits

Suppose that $f(x)$ is a function, and that a is a number so that for some number ϵ, $f(x)$ is defined whenever $a - \epsilon < x < a$ or $a < x < a + \epsilon$.

Then we defined (for some number L)

$$\lim_{x \longrightarrow a} f(x) = L$$

if as we make x closer and closer to a we can make $f(x)$ closer and closer to L.

Examples:

Calculate the following limits:

$$\lim_{x \to 0} x^2 + 3x - 5$$

$$\lim_{x \to 3} \frac{x-3}{\sqrt{x-3}}$$

$$\lim_{x \to 2} \frac{x-2}{\sqrt{x}-\sqrt{2}}$$

Continuity

A function $f(x)$ is **continuous at a** if $f(a)$ exists and

$$\lim_{x \to a} f(x) = f(a)$$

Continuity

A function $f(x)$ is **continuous at a** if $f(a)$ exists and

$$\lim_{x \to a} f(x) = f(a)$$

If a function is defined by a single formula then (for our purposes) it will be continuous wherever it is defined.

Continuity

A function $f(x)$ is **continuous at a** if $f(a)$ exists and

$$\lim_{x \to a} f(x) = f(a)$$

If a function is defined by a single formula then (for our purposes) it will be continuous wherever it is defined.

However, if it is defined in a 'piecewise' manner – like in the example of domains from earlier – then we have to work with one-sided limits.

Example:

Where is the following function continuous and where is it discontinuous?

$$f(x) = \begin{cases} x^2 - 3x - 5 & \text{if } x \leq -2 \\ 2x + 9 & \text{if } -2 < x < 0 \\ 7 & \text{if } x = 0 \\ \sqrt{x^2 + 2x + 9} & \text{if } 0 < x \end{cases}$$

The Derivative

The **derivative** of a function $f(x)$ at a point $x = a$ gives the infinitesimal rate of change of $f(x)$ at the point $x = a$. It is also the slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$.

The Derivative

The **derivative** of a function $f(x)$ at a point $x = a$ gives the infinitesimal rate of change of $f(x)$ at the point $x = a$. It is also the slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$.

We can define it formally as:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

The Derivative

The **derivative** of a function $f(x)$ at a point $x = a$ gives the infinitesimal rate of change of $f(x)$ at the point $x = a$. It is also the slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$.

We can define it formally as:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

We've seen various questions in lectures, homework and the first term where we had to calculate a derivative from the above definition, and there could be another one on the final.

Rules for differentiation

But usually, we differentiate using the various rules for derivatives:

- Constant Multiple Rule
- Power rule
- Sum Rule (and Difference Rule)
- Product Rule
- Quotient Rule
- Chain Rule
- Rules for differentiating logarithms and exponentials.

Honestly, if you don't know these rules by now, you're probably in trouble.

Honestly, if you don't know these rules by now, you're probably in trouble.

Would anyone like me to revise any particular rule?

The tangent line

Suppose that we have a function $f(x)$, and we consider its graph $y = f(x)$, and a point $x = a$. The **equation of the tangent line to $y = f(x)$ at $x = a$ is**

$$y = f'(a)x - \left[f'(a)a + f(a)\right]$$

The tangent line

Suppose that we have a function $f(x)$, and we consider its graph $y = f(x)$, and a point $x = a$. The **equation of the tangent line to $y = f(x)$ at $x = a$ is**

$$y = f'(a)x - \left[f'(a)a + f(a)\right]$$

We've also written this (in point-slope form) as

$$y - f(a) = f'(a)\,(x - a)$$

Approximation using the derivative: Marginal Analysis

The function given by the tangent line gives the linear function which is the best approximation to the function $f(x)$ amongst all linear functions (near the point $x = a$).

Therefore, we can use it to estimate values of functions, or the change in the value of functions.

Approximation using the derivative: Marginal Analysis

The function given by the tangent line gives the linear function which is the best approximation to the function $f(x)$ amongst all linear functions (near the point $x = a$).

Therefore, we can use it to estimate values of functions, or the change in the value of functions.

In economics, this estimation is called **marginal analysis.**

The relevant equations are

$$\Delta f \approx f'(a)\Delta x$$

(where Δf is the change in f values and Δx is the change in x values). Formally, if we're considering the change from $x = a$ to $x = b$ we have:

$$f(b) - f(a) \approx f'(a)(b - a).$$

The relevant equations are

$$\Delta f \approx f'(a)\Delta x$$

(where Δf is the change in f values and Δx is the change in x values). Formally, if we're considering the change from $x = a$ to $x = b$ we have:

$$f(b) - f(a) \approx f'(a)(b - a).$$

In terms of estimating values of functions we have:

$$f(b) \approx f(a) + f'(a)(b - a).$$

[Often, in examples, you'll be given a and the change $b-a$, rather than b explicitly.]

Examples:

(1) Suppose that $f(x) = x^3 + 3x^2 - 2x - 10$. Use the derivative to estimate $f(1.01)$, using information at $x = 1$.

(2) Suppose that in terms of input q units of labor, a production function is

$$C(q) = 0.01q^3 - 0.2q^2 + q + 1000.$$

(a) What is the production when $q = 50$?

(b) Use marginal analysis to estimate the increase in production when one extra unit of labor is used.

(c) What is the actual increase in production when one extra unit of labor is used?

Implicit Differentiation:

Implicit differentiation is used when we are given a relationship between x and y that is not of the form $y = f(x)$, but is rather an equation that x and y are supposed to satisfy.

When we use implicit differentiation, we are still treating y as a function of x, and so we use the Chain Rule when we differentiate things with a y in them.

Implicit Differentiation:

Implicit differentiation is used when we are given a relationship between x and y that is not of the form $y = f(x)$, but is rather an equation that x and y are supposed to satisfy.

When we use implicit differentiation, we are still treating y as a function of x, and so we use the Chain Rule when we differentiate things with a y in them.

Honestly, we didn't go much more into the details of how this works, and the best thing to do is to work a bunch of examples until it starts to kind of make sense to you. A thousand examples might suffice...

Implicit Differentiation:

Implicit differentiation is used when we are given a relationship between x and y that is not of the form $y = f(x)$, but is rather an equation that x and y are supposed to satisfy.

When we use implicit differentiation, we are still treating y as a function of x, and so we use the Chain Rule when we differentiate things with a y in them.

Honestly, we didn't go much more into the details of how this works, and the best thing to do is to work a bunch of examples until it starts to kind of make sense to you. A thousand examples might suffice...

Let's have a few examples to practice for Wednesday or Friday...

Examples:

In the following example, calculate $\frac{dy}{dx}$.

(1) $y^2 + xy - 3x = 0$

(2) $y^2 = x^3 - 3x + 5$

(3) $yx^2 - 2x^2 - xy + 5 = 0$

Analyzing the behavior of functions with derivatives

We now have lots of tools for describing the behavior of functions.

- The x-intercept occurs at points where $f(x) = 0$.

- The y-intercept occurs at the point $(0, f(0))$.

- The critical points of a function $f(x)$ are those points where $f'(x) = 0$ or $f'(x)$ is not defined.

- The Second Derivative Test can tell us whether these critical points are relative maxima or relative minima.

• Between critical points, the function is either increasing or decreasing, and the sign of the first derivative tells us which.

• The function is **concave upwards** if $f''(x) > 0$, and **concave downwards** if $f''(x) < 0$.

• The places where the concavity changes between concave upwards and concave downwards are called **points of inflection** (and they are points where $f''(x) = 0$ or $f''(x)$ is not defined).

• A **horizontal asymptote** occurs at $y = c$ when either

$$\lim_{x \to \infty} f(x) = c$$

or

$$\lim_{x \to -\infty} f(x) = c.$$

- A **vertical asymptote** at $x = a$ when either

$$\lim_{x \to a^-} f(x) = \pm\infty,$$

or

$$\lim_{x \to a^+} f(x) = \pm\infty.$$

Examples

(1) Describe the salient features of the graph of $y = x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$.

(2) Describe the salient features of the graph of $y = \frac{2x^2-1}{x^2+2}$

[Then sketch the graphs...]

Absolute maxima and minima of functions (on closed bounded intervals)

Suppose we are considering a function $f(x)$ on an interval $a \leq x \leq b$. To find the maximum and minimum values of $f(x)$ on this interval, we do the following:

Absolute maxima and minima of functions (on closed bounded intervals)

Suppose we are considering a function $f(x)$ on an interval $a \leq x \leq b$. To find the maximum and minimum values of $f(x)$ on this interval, we do the following:

• Find the critical points of $f(x)$ on the interval $a \leq x \leq b$. (These are the points where $f'(x) = 0$ or $f'(x)$ does not exist.

Absolute maxima and minima of functions (on closed bounded intervals)

Suppose we are considering a function $f(x)$ on an interval $a \leq x \leq b$. To find the maximum and minimum values of $f(x)$ on this interval, we do the following:

• Find the critical points of $f(x)$ on the interval $a \leq x \leq b$. (These are the points where $f'(x) = 0$ or $f'(x)$ does not exist.

• Find the values $f(a)$, $f(b)$ and $f(c)$ for each critical point c. The maximum value is the largest of these numbers, and the minimum the smallest of these numbers.

WARNING: If there is a number c where $f'(c)$ does not exist, it could be because $f(c)$ does not exist. If there is a vertical asymptote at $x = c$ then there might be no maximum and/or no minimum value for $f(x)$...

Examples

Find the maximum and minimum values of the following functions on the given intervals:

(1) $f(x) = \frac{2x^2-1}{x^2+2}$ for $-1 \leq x \leq 2$.

(2) $f(x) = x^3 - x^2 + 7$ for $0 \leq x \leq 3$.

Review to be continued...

(There are about ten more major topics...)