Concavity

Suppose that $f(x)$ is differentiable on an interval $a < x < b$. Then the graph of f is

• **concave upward** on $a < x < b$ if f' is **increasing** on the interval; and

• **concave downward** on $a < x < b$ if f' is **decreasing** on the interval.

Concave upward graphs lie *above* the tangent lines, and

concave downward graphs lie *below* the tangent lines.

PICTURES

Second derivative test for determining intervals of concavity

(1) Find points where $f''(x) = 0$ or $f''(x)$ does not exist. This breaks the domain up into a number of intervals.

(2) On each interval, check the sign of $f''(c)$, at some value c.

• If $f''(c) > 0$ then the graph of f is concave upward on the interval;

• If $f''(c) < 0$ then the graph of f is concave downward on the interval.

Points of inflection

A *point of inflection* of a function $f(x)$ is a point $(c, f(c))$ so that the concavity of the graph f changes at this point.

At a point of inflection $(c, f(c))$, we either have $f''(c) = 0$ or else $f''(c)$ does not exist.

[So, to find points of inflection, find the points as on the previous slide, and check that the sign of $f''(x)$ changes on either side of such a point.]

So, when we're sketching curves, we should also put in concavity and points of inflection...

[By the way, in economics, a point of inflection (when a function goes from concave upward to downward) is the point where the derivative is maximum, and is called the **point of diminishing returns**.]

Note also, that $(0,0)$ is not a point of inflection for $f(x) = x^4$. Why?

On the other hand, for $f''(x) = x^{1/3}$, the point $(0,0)$ is an inflection point, even though $f''(0)$ is not defined.

Examples

Find intervals of concavity, points of inflection (and everything else) about the following functions. Then sketch their graph.

- $f_1(x) = 3x - 5$
- $f_2(x) = 4 + 2x - x^2$
- $f_3(x) = x^3 - x^2 + 3x - 3$
- $f_4(x) = x^4 - 24x^2 + 36x - 44$