1. Equivalence Relations

1.1. **Sets.** A *set* is a collection of things. We won't be more precise than that for now.

If X is a set, the notation $x \in X$ means that x is an element of x.

If S is a set a *subset* of S is some collection of the elements of S, but maybe not all of them. Formally, we say that A is a subset of B if every element of A is an element of B. We write $A \subset B$.

We can write sets in various different ways. Here are some examples of ways of describing the same set ($\mathbb{Z}$ is the set of integers):

(1) $X = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$
(2) $X = \{2k \mid k \in \mathbb{Z}\}$
(3) $X = \{m \mid \text{ there exists an integer } k \text{ so that } m = 2k\}$
(4) X is the set of all even integers.

If A and B are sets then

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

is the set of all (ordered) pairs of elements, one from A and one from B.

1.2. **Relations.** Suppose that S is a set. A *relation* on S is a subset R of $S \times S$. Therefore, it is a collection of ordered pairs of elements in S. If $(a, b) \in R$ then we write $a \sim b$. We may refer to the relation as $\sim$.

1.3. **Examples.** Here are some relations on $\mathbb{Z}$, the set of integers.

(1) Let $x \sim y$ is $x + y$ is even.
(2) Let $a \sim b$ if $a \leq b$.
(3) Let $x \sim y$ if $x^2 + y^2 = 10$.
(4) Fix a natural number n, let $x \sim y$ if $n|(x - y)$.
(5) Let $r \sim s$ if $r < s$.

Exercise 1. *Make up some of your own relations on the integers.*

1.4. **Properties of relations.** Here are some properties the relations might have.

Suppose that $\sim$ is a relation on a set S.

The relation $\sim$ is *reflexive* if for all elements $a \in S$ we have $a \sim a$.

The relation $\sim$ is *symmetric* if for all elements $r, s \in S$, if $r \sim s$ then $s \sim r$.

The relation $\sim$ is *transitive* if for all elements $x, y, z \in S$, if $x \sim y$ and $y \sim z$ then $x \sim z$.

If a relation is reflexive, symmetric and transitive then we say that it is an *equivalence relation.*

Exercise 2. *For the relations listed above, and those that you invented, determine which of the three properties of symmetric, reflexive and transitive they satisfy. Say which are equivalence relations.*

1.5. **Equivalence classes.** Suppose that $\sim$ is an equivalence relation on a set X. Define the equivalence class of x, which we denote as $[x]$, by

$$[x] = \{y \in X \mid x \sim y\}.$$

We say that x is a *representative* of $[x]$.

WARNING: Because an equivalence class can have many elements, it can have many representatives.

Theorem 3. *Suppose that $\sim$ is an equivalence relation on a set S. Every element of S is contained in exactly one equivalence class.*

If two equivalence classes have any elements in common then they are identical. In other words, equivalence classes are either disjoint or equal.

We say that a collection of subsets of a set A which are so that every element of A is in exactly one of the subsets form a *partition* of A.

Example 4. *Suppose that $P = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$*

An example of a partition of P is:

$$\{1, 3\}, \{2, 5, 6, 7\}, \{4\}, \{8, 9\}.$$

1.6. **Applications.** Fix a natural number n, and suppose that n is at least 2.

Various propositions from the number theory worksheet show that the following relation is an equivalence relation:

Say $x \sim y$ if $x \equiv y \mod n$.

Exercise 5. *Which propositions do you need to show that this is an equivalence relation?*

It follows from Theorems 29 and 30 on the Number Theory worksheets that there are exactly n equivalence classes, which are $[0], [1], \ldots, [n-1]$.

Note, however, that $[2n] = [n] = [0] = [-n]$, which are all equally good ways of labelling this class.

Definition 6. *Suppose that $[x]$ and $[y]$ are equivalence classes for the relation $\sim$ (which is defined by $a \sim b$ if $n|(a-b)$. Then we define:*

$$[x] + [y] = [x + y]$$

and

$$[x].[y] = [x.y]$$

Exercise 7. *These operations are* well-defined. *This means that if different representatives are chosen for $[x]$ and $[y]$, the resulting classes on the right are the same class.*

Be clear about what this means, make sure you know how to prove it (by referencing some propositions on the number theory worksheet.