Functions and relations

Definition 1. *Suppose that* X *and* Y *are sets. A* function f *from* X *to* Y *is a way that assigns to each element of* $x \in X$ *a unique element* $f(x) \in Y$. *We write* $f : X \to Y$.

Remark 2. *Two functions* $f : X \to Y$ *and* $g : X \to Y$ *are the same function if they have the same values. In other words, they are the same if for every* $x \in X$ *we have* $f(x) = g(x)$.

However, if we have functions $f : X_1 \to Y_1$ *and* $g : X_2 \to Y_2$ *so that either* $X_1 \neq X_2$ *or* $Y_1 \neq Y_2$ *then the functions* f *and* g cannot *be the same.*

Notation 3. *One way of specifying a function is by the pairs* $(x, f(x))$, *as elements of* $X \times Y$. *A* relation *is a subset of* $X \times Y$, *and so a function from* X *to* Y *defines a particular kind of relation on* $X \times Y$.

Question 4. *Can you give a condition on a relation that says* exactly *when it comes from a function as above?*

Question 5. *Suppose that* X *is the set* $\{a, b, c\}$ *and* Y *is the set* $\{10, 13, 22, 87\}$, *which of the following sets defines a function from* X *to* Y? *Which of them are relations on* X *and* Y *that are not functions?*

1. $\{a, 10, b, 13\}$
2. $\{(a, 10)\}$
3. $\{(a, 10), (b, 10), (c, 10)\}$
4. $\{(a, 10), (a, 13), (a, 22)\}$
5. $\{(a, b), (10, c)\}$
6. $\{(a, 87), (c, 13), (b, 22)\}$
7. $\{(a, 12), (b, 13), (c, 87)\}$
8. $\{(a, 13), (b, 13)\}$

.

Definition 6. *Suppose that* X, Y *and* Z *are sets, and that* $f : X \to Y$ *and* $g : Y \to Z$ *are functions. We define a new function* $g \circ f : X \to Z$ *by*

$$g \circ f(x) = g(f(x)).$$

Proposition 7. *Suppose that* $f : X \to Y$ *and* $g : Y \to Z$ *are functions. Prove that* $g \circ f : X \to Z$ *is a function.*

[In other words, prove that the previous definition does indeed define a function.

Definition 8. *A function $f : X \to Y$ is an* injection *if whenever x_1, x_2 satisfy $f(x_1) = f(x_2)$, we have $x_1 = x_2$.*

Definition 9. *A function $f : X \to Y$ is a* surjection *if for any y in Y there exists at least one $x \in X$ so that $f(x) = y$.*

Definition 10. *A function $f : X \to Y$ is a* bijection *if it is an injection and a surjection.*

Exercise 11. *Let $X = \{1, 2, 3, 4, 5, 6, 7\}$ and $Y = \{2, 3, 5, 7, 11, 13, 17\}$. Write down two examples of each of the following kinds of functions:*

1. *A bijection*
2. *A function which is neither an injection nor a surjection*
3. *An injection which is not a surjection*
4. *A surjection which is not an injection*

Definition 12. *Suppose that X is a set. The* identity function *$I_X : X \to X$ is the function defined by $I_X(x) = x$ for all $x \in X$.*

Lemma 13. *Let X be a set. Then the identity function $I_X : X \to X$ is a bijection.*

Proposition 14. *Suppose that $f : X \to Y$ is a bijection. There is a unique function $g : Y \to X$ so that $g \circ f = I_X$.*

For this function, we also have $f \circ g = I_Y$.

The function $g : Y \to X$ in Proposition 14 is called the *inverse* of f.

Theorem 15. *Let m and n be natural numbers. Suppose that X is a set with m elements in it and Y is a set which has n elements in it.*

1. *If there is an injection from X to Y then $m \le n$.*
2. *If there is a surjection from X to Y then $m \ge n$.*
3. *If there is a bijection from X to Y then $m = n$.*

Question 16. *If sets X and Y have infinitely many elements, is there always a bijection between X and Y?*

Theorem 17. *Let X and Y be sets and suppose that $f : X \to Y$ is a function. Define a relation R_f on X by saying that $(x_1, x_2) \in R_f$ if $f(x_1) = f(x_2)$. Prove that R_f is an equivalence relation on X.*

Definition 18. *Suppose that $\mathbb{F}$ is a field. A function $f : \mathbb{F} \to \mathbb{F}$ is called* linear *if*

1. *For all $x, y \in \mathbb{F}$ we have $f(x + y) = f(x) + f(y)$;*
2. *For all $x, y \in \mathbb{F}$ we have $f(xy) = xf(y)$.*

Exercise 19. *Suppose that we consider the field $\mathbb{R}$ of real numbers. Which of the following are linear functions?*

1. $f(x) = x^2$;
2. $g(x) = 0$;
3. $h(x) = 2x + 3$;
4. $j(x) = x$;
5. $k(x) = \pi x$;
6. $l(x) = \sin(x)$.

Exercise 20. *Suppose that $\mathbb{F}$ is a field. Describe all of the linear functions from $\mathbb{F}$ to $\mathbb{F}$. Prove that your answer is correct (meaning that all the functions you list are linear and all linear functions are on your list).*