Graded Homework 3: Due Monday October 30, 2017 at the beginning of class

1. (8pts) Let n be a natural number bigger than 1 and suppose that $\mathbf{Z}_n$ is the set of equivalence classes of integers modulo n.

   (a) Prove that Axioms (A4), (M1) and (D1) from the Fields worksheet hold (with the definitions of addition and multiplication we gave in class – note that these definitions are also on the worksheet on equivalence relations).

   (b) Prove that Axiom (M4) holds when $n = 5$. Explain why it doesn't hold when $n = 6$.

2. (9pts) Let $S = \{w, x, y, z\}$ be a set, and suppose that we have operations of addition and multiplication on S given by the following tables.

$+$	w	x	y	z
w	z	y	x	w
x	y	z	w	x
y	x	w	z	y
z	w	x	y	z

$.$	w	x	y	z
w	x	y	w	z
x	y	w	x	z
y	w	x	y	z
z	z	z	z	z

   (a) Define an element 0 of S and verify that Axiom (A2) holds.

   (b) For each element a of S define an element $-a$, and verify that Axiom (A3) holds (with the choice of 0 you made in the previous part).

   (c) Define an element 1 of S and verify that Axiom (M2) holds. Make sure (so that (NT1) holds) that you choose a different element than the one you chose for 0.

   (d) For each element a of S (other than the 0 from the first part), define an element a^{-1} and verify that Axiom (M4) holds.

   [You may assume that the remaining axioms hold without proving them. If it turns out that they don't, you get many bonus points if you explain to me the mistake I made, so I can fix it. Please note that this example is **NOT** the integers mod 4 in disguise, it is a different kind of example.]

3. (8pts) Let X be the set $\mathbb{Z} \times \mathbb{N}$ consisting of pairs (u, v) where u is an integer and v is a natural number. Define a relation on X by $(x, y) \sim (r, s)$ if $xs - yr = 0$.

(a) Prove that this is an equivalence relation.

(b) Prove that in each equivalence class $[(a, b)]$ there is a unique element (p, q) so that for any $(x, y) \in [(a, b)]$ we have $q \leq y$.

(c) Fix an equivalence class $[(k, l)]$, and find the element (f, g) as in the previous part (where the second element is smallest). Prove that

$$[(k, l)] = \{(fn, gn) \mid n \in \mathbb{N}\}.$$