Math 210 Exercise Problems 4/7/09

14.4. #19

Let $(a, b, f(a,b))$ be the pt. on the graph of

$z = 3x^2 - 4y^2$. , $f(x,y) = 3x^2 - 4y^2$.

the tangent plane at $(a, b, f(a,b))$ is:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$f_x(a,b)(x-a) + f_y(a,b)(y-b) - z = -f(a,b)$$

thus the normal vector to the tangent plane.

$\vec{n} = \langle f_x(a,b), f_y(a,b), -1 \rangle$

So. $\dfrac{f_x(a,b)}{3} = \dfrac{f_y(a,b)}{2} = \dfrac{-1}{2}$

$\Rightarrow f_x(a,b) = -\frac{3}{2}$

$f_y(a,b) = -1$

on the other hand, $f(x,y) = 3x^2 - 4y^2$.

$f_x(x,y) = 6x$

$f_y(x,y) = -8y$

$\therefore f_x(a,b) = 6a$

$f_y(a,b) = -8b$

so, we have $6a = -\frac{3}{2}$, $-8b = -1$, solving for a, b

$a = -\frac{1}{4}$, $b = \frac{1}{8}$

thus the pt. is $(-\frac{1}{4}, \frac{1}{8}, f(-\frac{1}{4}, \frac{1}{8}))$

14.5. #30

$$f(x, y, z) = \sin(xy + z), \quad P = (0, -1, \pi)$$

$$D_u f(P) = \nabla f_P \cdot u$$

$$= \|\nabla f_P\| \, \|u\| \cos\theta$$

$$= \|\nabla f_P\| \cos\theta \quad \text{where } \theta = 30^\circ$$

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

$$= \langle \cos(xy+z) \cdot y, \ \cos(xy+z)x, \ \cos(xy+z) \rangle$$

$$\nabla f_P = \langle \cos(0+\pi)(-1), \ \cos(0+\pi)0, \ \cos(0+\pi) \rangle$$

$$= \langle 1, 0, -1 \rangle$$

$$\|\nabla f_P\| = \sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$$

$$\therefore D_u f(P) = \sqrt{2} \cos(30^\circ)$$

$$= \sqrt{2} \cdot \frac{\sqrt{3}}{2}$$

$$= \sqrt{6}/2$$

-.5. #42.

$xz + 2x^2y + y^2z^3 = 11$, $P = (2,1,1)$

the ~~f~~ surface

$F(x,y,z) = xz + 2x^2y + y^2z^3 = 11$

the eqn. of the tangent plane to $F(x,y,z) = 11$ at $P = (a,b$ is.

$$\nabla F_P \cdot \langle x-a, y-b, z-c \rangle = 0$$

$$\nabla F = \langle F_x, F_y, F_z \rangle$$

$$= \langle z + 4xy + 0,\ 2x^2 + \overset{2yz^3}{\cancel{z^3}},\ x + \overset{3y^2z^2}{\cancel{y^2}} \rangle$$

$$\nabla F_P = \langle 1 + 4 \cdot 2 \cdot 1 + 0,\ 2 \cdot 2^2 + \overset{2 \cdot 2}{1^3},\ 2 + \overset{3 \cdot 1^2}{1^2} \rangle$$

$$= \langle 9, \overset{12}{\cancel{9}}, \overset{5}{\cancel{3}} \rangle$$

∴, the tangent plane is

$$\langle 9, \overset{12}{\cancel{9}}, \overset{5}{\cancel{3}} \rangle \cdot \langle x-2, y-1, z-1 \rangle = 0$$

$$9(x-2) + \overset{12}{\cancel{9}}(y-1) + \overset{5}{\cancel{3}}(z-1) = 0$$

$$9x + \overset{12}{\cancel{9}}y + \overset{5}{\cancel{3}}z = \overset{35}{\cancel{30}}$$

$$\boxed{\cancel{3x + 3y + z = 10}}$$

X

14.6. #13

$\frac{\partial g}{\partial \theta}$ at $(r, \theta) = (2\sqrt{2}, \frac{\pi}{4})$

where $g(x, y) = \frac{1}{x + y^2}$

(r, θ) polar coordinates

$$\frac{\partial g}{\partial \theta} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= \frac{-1}{(x+y^2)^2} \cdot r(-\sin\theta) + \frac{-1 \cdot 2y}{(x+y^2)^2} \cdot r\cos\theta.$$

simplify it., represent the answer in r, θ.

then evaluate at $(2\sqrt{2}, \frac{\pi}{4})$

HW7

14.7. $\cancel{14.8}$

4.7. #13 $f(x,y)=xye^{-x^2-y^2}=(xe^{-x^2})(ye^{-y^2})$

$f_x=e^{-x^2}(ye^{-y^2})-2x^2e^{-x^2}(ye^{-y^2})$

$=(ye^{-y^2})(e^{-x^2}-2x^2e^{-x^2})$

$=(ye^{-y^2})e^{-x^2}(1-2x^2)$

$=0$

$y=0,\ 1-2x^2=0 \Rightarrow x^2=\frac{1}{2} \Rightarrow x=\pm\frac{\sqrt{2}}{2}$

$f_y=(xe^{-x^2})e^{-y^2}(1-2y^2)=0$

$x=0,\ 1-2y^2=0 \Rightarrow y=\pm\frac{\sqrt{2}}{2}$

critical pts: $(0,0)$ $(\frac{\sqrt{2}}{2},\pm\frac{\sqrt{2}}{2})$. $(-\frac{\sqrt{2}}{2},\pm\frac{\sqrt{2}}{2})$

$f_{xx}=(ye^{-y^2})(-2xe^{-x^2})(1-2x^2)+(ye^{-y^2})e^{-x^2}(-4x)$

$=(ye^{-y^2})e^{-x^2}[(-2x)(1-2x^2)-4x]$

$=(ye^{-y^2})e^{-x^2}[-2x+4x^3-4x]$

$=(ye^{-y^2})e^{-x^2}(4x^3-6x)$

$=-2xye^{-y^2-x^2}(3-2x^2)$

$f_{yy}=-2xye^{-y^2-x^2}(3-2y^2)$

$f_{xy}=e^{-x^2}(1-2x^2)[e^{-y^2}-2y^2e^{-y^2}]$

$=e^{-x^2-y^2}(1-2x^2)(1-2y^2)$

$D=f_{xx}f_{yy}-f_{xy}^2$

$=(3-2x^2)(3-2y^2)(4x^2y^2e^{-2y^2-2x^2})-(1-2x^2)(1-2y^2)e^{-2x^2-2y^2}$

c.p.	f_{xx}	f_{yy}	f_{xy}	D	Type
$(0, 0)$	0	0	1	-1	saddle
$(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$	$-\frac{2}{e}$	$-\frac{2}{e}$	0	$\frac{4}{e^2}$	local max
$(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$	$\frac{2}{e}$	$\frac{2}{e}$	0	$\frac{4}{e^2}$	local min
$(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$	$\frac{2}{e}$	$\frac{2}{e}$	0	$\frac{4}{e^2}$	local min
$(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$	$\frac{-2}{e}$	$\frac{-2}{e}$	0	$\frac{4}{e^2}$	local max

14.8. #5 ~~$f(a,b)$~~

$f(x,y) = xy$ subject to ~~to~~ $g(x,y) = e^x - y = 0$

lagrange eqns :

$\nabla f = \lambda \nabla g$

$\langle y, x \rangle = \lambda \langle e^x, -1 \rangle \Rightarrow \begin{matrix} y = \lambda e^x \Rightarrow \lambda = \frac{y}{e^x} \\ x = -\lambda \end{matrix}$

$\therefore \frac{y}{e^x} = -x \Rightarrow y = -xe^x$

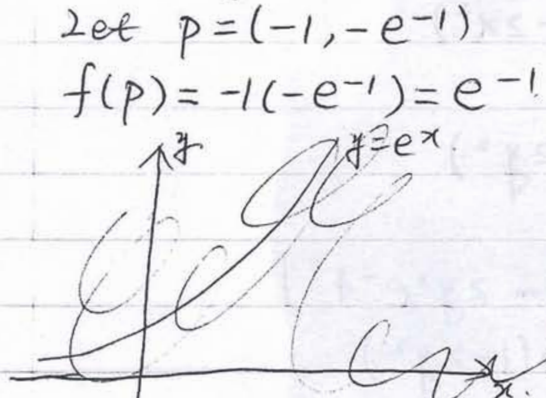
solve for x and y using the constraint $g(x,y)$

$g(x,y) = e^x + xe^x = 0 \Rightarrow x = -1$

thus, $y = -xe^x = -1e^{-1} = -e^{-1}$

Let $p = (-1, -e^{-1})$

$f(p) = -1(-e^{-1}) = e^{-1}$

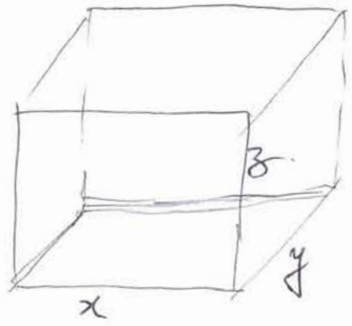


$f(x, e^x) = xe^x$. for $x > 0$. $xe^x > 0$

$\lim_{x \to -\infty} xe^x = \lim_{x \to -\infty} \frac{x}{e^{-x}} \overset{H}{=} \lim_{x \to -\infty} \frac{1}{-e^{-x}} = 0$

thus, we know $(-1, -e^{-1})$.

.8 #16.



Let x, y, z be the dimensions of the box

$f(x,y,z) = xyz$ Subject to $g(x,y,z) = 4x+4y+4z = 300$
the volume
i.e. $g(x,y,z) = x+y+z = 75$
$x \geq 0,\ y \geq 0,\ z \geq 0$

step 1 : write out the Lagrange Equations

$$\nabla f = \lambda \Delta g$$
$$\langle yz, xz, xy\rangle = \lambda \langle 1, 1, 1\rangle$$
$$\begin{cases} yz = \lambda \\ xz = \lambda \\ xy = \lambda \end{cases}$$

step 2 : solve for λ in terms of x, y and z.

$$\lambda = yz = xz = xy.$$

step 3 solve for x, y and z using the constraint

$$yz = xz \Rightarrow z(x-y) = 0$$
$$xy = xz \quad x(z-y) = 0$$

here, $x \neq 0,\ y \neq 0,\ z \neq 0$ otherwise the volume $= 0$

$\therefore x = y$
$z = y$ (*)

Substitute (*) into $g(x,y,z)=0$.

we have $x+x+x=75$

$$x=\frac{75}{3}=25$$

∴ the critical pt is $(25, 25, 25)$

step 4. make conclusion.

$V=f(x,y,z)=xyz=25^3 \text{ cm}^3$

$f(x,y,z)$ is a continuous function.

on the domain $\{(x,y,z) \mid x+y+z=75,\ x\geq 0,\ y\geq 0,\ z\geq 0\}$.

which is closed and bounded.

thus $f(x,y,z)$ has global extreme.

the ~~min~~ minimum value is 0.

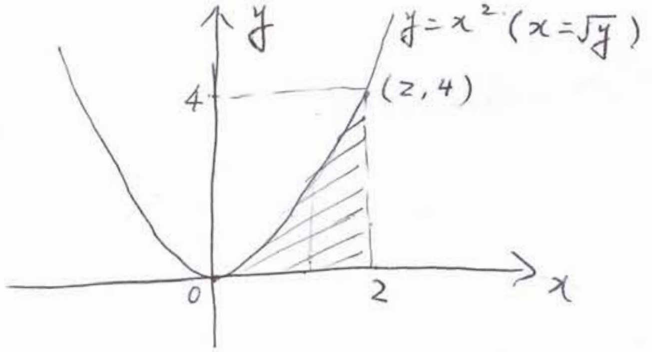
the max. value is 25^3.

So the box with max. value is a Cube of edge 25 cm.

5.2 #36.

$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^2+y}\, dx\, dy.$$

D: $\sqrt{y} \le x \le 2.\quad 0 \le y \le 4$



the domain D can be described as vertically simple region.

D: $0 \le x \le 2, \quad 0 \le y \le x^2$

$$\int_0^2 \int_{y=0}^{x^2} \sqrt{x^2+y}\, dy\, dx.$$

$$= \int_0^2 \int_{y=0}^{x^2} (x^2+y)^{\frac{1}{2}}\, dy\, dx$$

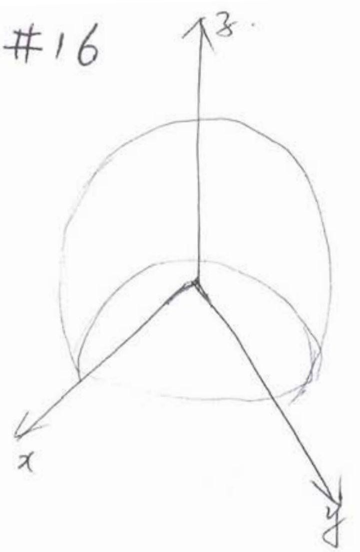
$$= \int_0^2 \frac{2}{3} (x^2+y)^{3/2} \Big|_{y=0}^{x^2} dx.$$

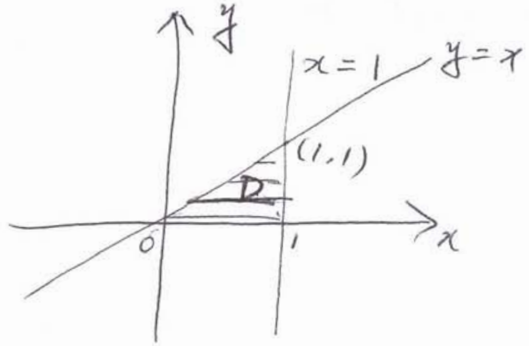
$$= \int_0^2 \frac{4\sqrt{2}-2}{3} x^3\, dx.$$

$$= \frac{4\sqrt{2}-2}{3} \cdot \frac{x^4}{4} \Big|_0^2$$

$$= \frac{16\sqrt{2}-8}{3}$$

15.3 #16





the upper surface is the hemisphere $z=\sqrt{9-x^2-y^2}$

the lower surface is the xy-plane $z=0$

the projection of W onto the xy-plane is the triangle D.

$$\iiint_W f(x,y,z)\,dV$$

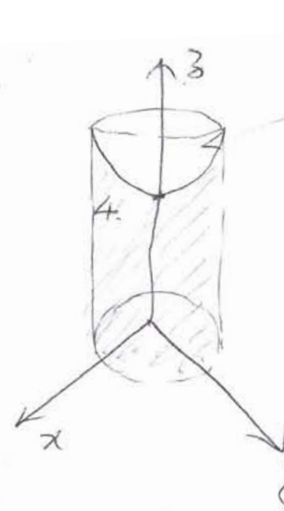
$$= \iint_D \left(\int_0^{\sqrt{9-x^2-y^2}} z\,dz\right) dA$$

$$= \int_0^1 \int_{x=y}^{1} \left(\int_0^{\sqrt{9-x^2-y^2}} z\,dz\right) dx\,dy$$

$$\vdots$$

$$= 2\frac{1}{12}$$

15.4 #30



elliptic paraboloid.

$z = 4 + x^2 + y^2$

the upper surface is $z = 4 + x^2 + y^2$
$= 4 + r^2$.

the lower surface is $z = 0$

the projection of w onto the xy-plane is D $x^2 + y^2 \le 1$ i.e. $r^2 \le 1$

$W: \quad 0 \le r \le 1, \quad 0 \le \theta \le 2\pi. \quad 0 \le z \le 4 + r^2.$

$$\iiint_W f(x, y, z)\, dV$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=0}^{4+r^2} z \, \underline{r\,dr\,d\theta}$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \frac{z^2 r}{2} \Big|_{z=0}^{4+r^2} dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} \frac{(4+r^2)^2 r}{2}\, dr\,d\theta$$

Let $u = 4 + r^2$, $du = 2r\,dr. \Rightarrow r\,dr = \frac{du}{2}$

$$= \int_{\theta=0}^{2\pi} \int_{4}^{5} \frac{u^2}{2} \cdot \frac{du}{2}\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{u^3}{12} \Big|_4^5 d\theta$$

$$= \int_{\theta=0}^{2\pi} \frac{61}{12}\, d\theta$$

$$= \frac{61\pi}{6}.$$

15.5. #30.

$$R = [1,4] \times [1,4]$$

$$\Phi(u,v) = \left(\frac{u^2}{v}, \frac{v^2}{u}\right)$$

① Compute $\mathrm{Jac}(\Phi)$

$$\mathrm{Jac}(\Phi) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{2u}{v} & \frac{-u^2}{v^2} \\ -\frac{v^2}{u^2} & \frac{2v}{u} \end{vmatrix}$$

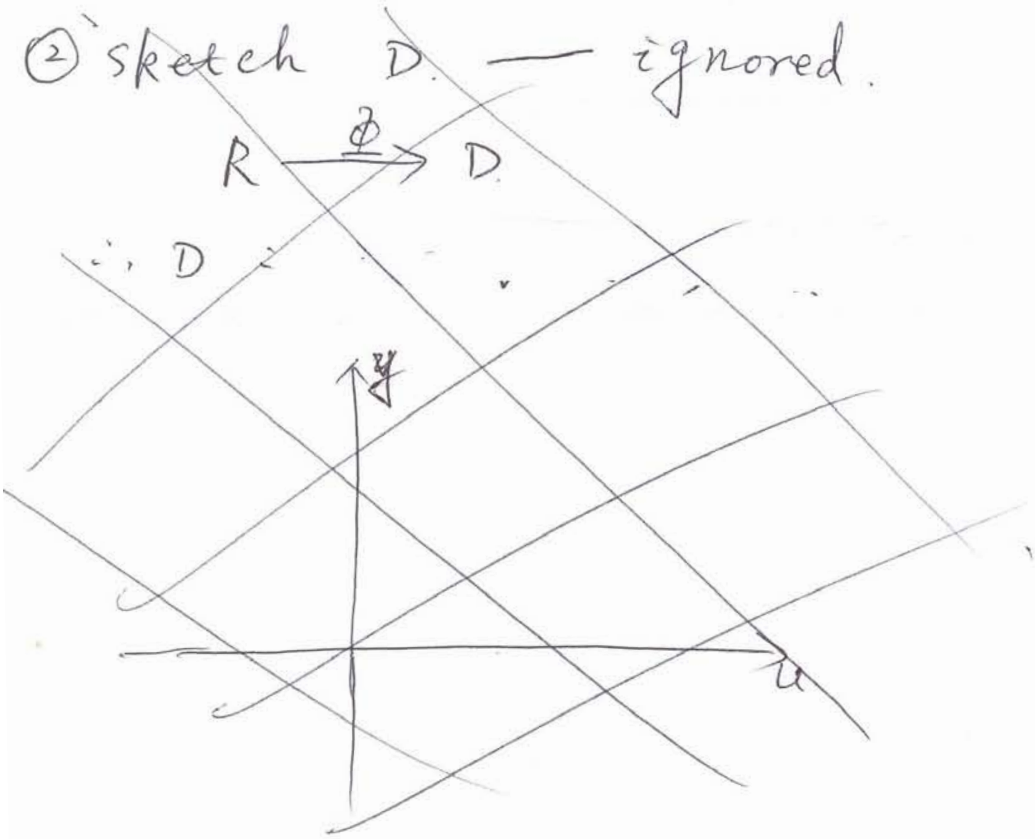
$$= \frac{2u}{v} \cdot \frac{2v}{u} - \left(\frac{-u^2}{v^2}\right)\left(-\frac{v^2}{u^2}\right)$$

$$= 3$$

② sketch D — ignored.

$$R \xrightarrow{\Phi} D$$

$\therefore$ D =



(3) ~~$f(x,y) = x^2+y^2$~~

Since $x = \frac{u^2}{v}$. $y = \frac{v^2}{u}$

$$f(x,y) = \left(\frac{u^2}{v}\right)^2 + \left(\frac{v^2}{u}\right)^2 = \frac{u^4}{v^2} + \frac{v^4}{u^2}$$

Using change of variables

$$\iint_D f(x,y)\,dA = \iint_R f(x(u,v), y(u,v))\,|Jac(\Phi)|\,du\,dv$$

$$= \int_1^4 \int_1^4 \left(\frac{u^4}{v^2} + \frac{v^4}{u^2}\right) \cdot 3\,du\,dv.$$

$$= \cdots$$

$$= 920.7$$