On the Characteristic Classes of PU_n and Its Application

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Overview

Classifying Spaces

Let G be a topological group. Then its classifying space $\mathbf{B}G$, equipped with a base point ω_0, is characterised, up to homotopy, by the following property:

For any finite CW-complex X with a base point x_0, the set of isomorphism classes of principal G-bundles is isomorphic to $[X; x_0, \mathbf{B}G; \omega_0]$, the set of pointed homotopy classes of maps from (X, x_0) to $(\mathbf{B}G, \omega_0)$.

Topologists like to study the cohomology of classifying spaces.

Example

Let U_n, O_n be the unitary group and orthogonal group of degree n, respectively. Then

- $H^*(\mathbf{B}U_n;\mathbb{Z}) \cong \mathbb{Z}[c_1,\cdots,c_n]$, where c_i is the ith universal Chern class.
- $H^*(\mathbf{B}O_n;\mathbb{Z}/2) \cong \mathbb{Z}[w_1,\cdots,w_n]$, where w_i is the ith universal Stiefel-Whitney class.

Example

Consider the inclusion $S^1 \hookrightarrow U_n$ of scalar matrices. The quotient group PU_n is called the *projective unitary group* of degree n.

$$H^*(\mathbf{B}PU_n; \mathbb{Z}) \cong ???$$

Not much is known about the cohomology of $\mathbf{B}PU_n$.

- It is easy to find $H^i(\mathbf{B}PU_n;\mathbb{Z}) = 0$ for $i = 1, 2$, $H^3(\mathbf{B}PU_n;\mathbb{Z}/n) \cong \mathbb{Z}/n$.
- A. Kono and M. Mimura found the $\mathbb{Z}/2$-module structure of $H^*(\mathbf{B}PU_{4k+2};\mathbb{Z}/2)$ (1971, [5]).
- L. Woodward found $H^4(\mathbf{B}PU_n;\mathbb{Z}) = \mathbb{Z}$ (1982, [7])
- A. Vavpetič and A. Viruel studied $H^*(\mathbf{B}PU_p;\mathbb{Z}/p)$ for a prime number p (2005, [6]).
- B. Antieau and B. Williams found $H^5(\mathbf{B}PU_p;\mathbb{Z}/n) = 0$ (2014, [1])

Theorem (X. G, [4], 2016)

For an integer $n > 1$, the graded ring $H^*(\mathbf{B}PU_n;\mathbb{Z})$, in degrees ≤ 10, is isomorphic to the following graded ring:

$$\mathbb{Z}[e_2, \cdots, e_{j_n}, x_1, y_{3,0}, y_{2,1}, z_1, z_2]/I_n.$$

Here e_i is of degree $2i$, $j_n = min\{5, n\}$; the degrees of $x_1, y_{3,0}, y_{2,1}$ are $3, 8, 10$, respectively; and the degrees of z_1, z_2 are $9, 10$, respectively. I_n is the ideal generated by

$$nx_1, \quad gcd\{2,n\}x_1^2, \quad gcd\{3,n\}y_{3,0}, \quad gcd\{2,n\}y_{2,1}, \quad gcd\{3,n\}z_1,$$

$$gcd\{3,n\}z_2, \quad \delta(n)e_2x_1, \quad (\delta(n)-1)(y_{2,1}-e_2x_1^2), \quad e_3x_1,$$

where

$$\delta(n) = \begin{cases} 2, & \textit{if } n = 4l+2 \textit{ for some integer } l, \\ 1, & \textit{otherwise.} \end{cases}$$

For example, $H^6(\mathbf{B}PU_n;\mathbb{Z})$ is isomorphic to $\mathbb{Z}\oplus\mathbb{Z}/2$ when $n>2$ is even, and $\mathbb{Z}$ if n is odd; $H^7(\mathbf{B}PU_n;\mathbb{Z})$ is isomorphic to $\mathbb{Z}/2$ when $n=4k+2$ for some integer k, and 0 otherwise.

Some interesting properties of $(\mathbf{B}G, \omega_0)$:

- $(G, 1) \simeq (\Omega\mathbf{B}G, c_{\omega_0})$
- If G is an Abelian topological group, then so is $\mathbf{B}G$, up to homotopy.

Remark

Though $\mathbf{B}G$ is defined only up to homotopy, there are homotopy theoretical techniques that makes the operation $\mathbf{B}(-)$ a functor from the category of topological groups in to certain category of topological spaces with additional structures. In this talk, we simply think of $\mathbf{B}(-)$ as a functor into topological spaces.

Eilenberg-Mac Lane Spaces

Definition

The Eilenberg-Mac Lane Space $K(\Pi, n)$ is characterized up to homotopy by the following property:

$$\pi_i(K(\Pi, n)) = \begin{cases} 0, i \neq n, \\ \Pi, i = n. \end{cases} \tag{1}$$

Remark

When Π is Abelian, we can easily prove by induction that $K(\Pi, n)$ is an Abelian topological group. Moreover, we have $\mathbf{B}K(\Pi, n) \simeq K(\Pi, n+1)$ and $K(\Pi, n) \simeq \Omega K(\Pi, n+1)$, where ΩX is the loop space of the topological space X with a specified base point.

Serre Fibration/Fiber Sequence

Definition

A map of based topological spaces $p : E \to B$ is called a Serre fibration if for any CW-complex X and the unit interval $I = [0, 1]$, the dashed arrow in the following commutative diagram always exists:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \times I \arrow[r] \arrow[ur, dotted] & B \end{tikzcd} \tag{2}$$

Let F be the fiber of p over the base point of B. Then $F \to E \to B$ is called a (Serre) fiber sequence.

Example

1. A short exact sequence of topological groups is a fiber sequence.
2. Let $U \to V \xrightarrow{\varphi} W$ be a fiber sequence such that U, V, W are topological groups and that the arrows are continuous homomorphisms, then $\mathbf{B}U \to \mathbf{B}V \to \mathbf{B}W$ is a fiber sequence.
3. In (2), if U is Abelian, then there is a fiber sequence $V \xrightarrow{\varphi} W \to \mathbf{B}U$.
4. Let X be a space with a based point x_0. Let $\mathbf{P}X$ be the space of in X ending with x_0. Then $\mathbf{P}X$ is contractible. Let $p : \mathbf{P}X \to X$ be the evaluation map at initial points. Then $\mathbf{P}X \xrightarrow{p} X$ is a Serre fibration with fiber over x_0 ΩX, and $\Omega X \to \mathbf{P}X \xrightarrow{p} X$ is a fiber sequence.

Serre Spectral Sequence

For any fiber sequence $F \to Y \to B$ such that B is simply connected, and any commutative ring R wiht a unit, we have the Serre spectral sequence $E_*^{*,*}$ with $E_2^{p,q} \cong H^p(B; H^q(F; R))$ which converges to $H^*(Y; R)$. There is also a similar homological version of the Serre spectral sequence. The operation of taking the Serre spectral sequence is a functor from the category of fiber sequences and commutative diagram of fiber sequences to the one of spectral sequences with suitable bi-degree of differentials.

Consider the short exact sequence of Lie groups

$$1 \to S^1 \to U_n \to PU_n \to 1.$$

By Example (1), (2) as above, we have the following fiber sequence

$$\mathbf{B}S^1 \to \mathbf{B}U_n \to \mathbf{B}PU_n.$$

$S^1 \simeq K(\mathbf{Z}, 1)$ is an Abelian topological group, and so is $\mathbf{B}^2 S^1 \simeq K(\mathbb{Z}, 2)$. By Example (3) as above, we have a fiber sequence

$$\mathbf{B}U_n \to \mathbf{B}PU_n \to K(\mathbb{Z}, 3). \tag{3}$$

Consider the maximal tori T^n, PT^n of U_n and PU_n, respectively, and we obtain a similar fiber sequence as (3) as follows:

$$\mathbf{B}T_n \to \mathbf{B}PT_n \to K(\mathbb{Z},3). \tag{4}$$

Finally, by Example (4) we have the following fiber sequence:

$$\begin{tikzcd}
K(\mathbb{Z},2) \arrow[r] \arrow[d, "\simeq"] & PK(\mathbb{Z},3) \arrow[r] \arrow[d, "\simeq"] & K(\mathbb{Z},3) \\
\mathbf{B}S^1 & \mathrm{pt} &
\end{tikzcd} \tag{5}$$

The Device for Computation

The fiber sequences (3), (4) and (5) together forms the following (up to homotopy) commutative diagram:

$$\begin{tikzcd}
\mathbf{B}S^1 \arrow[r] \arrow[d, "\mathbf{B}\Delta"] & \mathrm{pt} \arrow[r] \arrow[d] & K(\mathbb{Z},3) \arrow[d, "="] \\
\mathbf{B}T^n \arrow[r] \arrow[d] & \mathbf{B}PT^n \arrow[r] \arrow[d] & K(\mathbb{Z},3) \arrow[d, "="] \\
\mathbf{B}U_n \arrow[r] & \mathbf{B}PU_n \arrow[r] & K(\mathbb{Z},3)
\end{tikzcd} \tag{6}$$

where $\mathbf{B}\Delta$ is induced by the inclusion of a scalar into diagonal matrices: $\Delta : S^1 \to T^n$ and all the other arrows being the obvious ones.

Remark

The "up to homotopy" condition does not suffice, since replacing a space in a fiber sequence by a homotopy equivalent one does not necessarily produce another fiber sequence. However, modern homotopy theory, for example, the theory of model category or infinity category, allows one to formalize such unprecise thoughts in such a way that we are free to replace homotopy equivalences by isomorphisms, at least in the context of this talk.

Let ${}^U E_*^{*,*}, {}^T E_*^{*,*}, {}^K E_*^{*,*}$ be the Serre spectral sequences induced by the fiber sequences (3), (4), (5), respectively. Then the diagram 6 induces morphisms of spectral sequences ${}^K E_*^{*,*} \to {}^T E_*^{*,*} \to {}^U E_*^{*,*}$.

${}^{K}E_*^{*,*}$ and ${}^{T}E_*^{*,*}$

The spectral sequence ${}^{K}E_*^{*,*}$ is induced by the fiber sequence $K(\mathbb{Z},2) \to \mathrm{pt} \to K(\mathbb{Z},3)$, which has an algebraic model constructed and well understood in [4], based on the work of H. Cartan and J. P. Serre ([2]).
The spectral sequence ${}^{T}E_*^{*,*}$ is induced by the fiber sequence $\mathbf{B}T^n \to \mathbf{B}PT^n \to K(\mathbb{Z},3)$. ${}^{T}E_*^{*,*}$ is well understood based on ${}^{K}E_*^{*,*}$, since $\mathbf{B}T^n \simeq (\mathbf{B}S^1)^n \simeq (K(\mathbb{Z},2))^n$.

To relate ${}^{T}E_*^{*,*}$ and ${}^{U}E_*^{*,*}$, we recall the following theorem well known to topologists.

Theorem

For any commutative ring R with a unit, $H^*(\mathbf{B}T^n;R) \cong R[v_1,\cdots,v_n]$ where v_i is of degree 2. $H^*(\mathbf{B}T^n;R) \cong R[c_1,\cdots,c_n]$ where c_i, the ith universal Chern class is of degree $2i$. Moreover the inclusion of maximal torus $T^n \hookrightarrow U_n$ induces a monomorphism $H^*(\mathbf{B}U_n;R) \to H^*(\mathbf{B}T^n;R)$ that sends c_i to σ_i, the ith elementary symmetric polynomial in $v_1,\cdots,v_n$.

A considerable number of differentials are determined by ${}^T E_*^{*,*}$ and the following

Theorem (X. G, [4], 2016)

The differentials of ${}^T E_*^{*,*}$ determine all of the differentials ${}^U d_r^{s-r,t+r-1}$ of ${}^U E_*^{*,*}$ such that for any $r' < r$, ${}^T d_{r'}^{s-r',t-r'+1} = 0$, by restricting to symmetric polynomials in $v_1, \cdots, v_n$.

With a little luck, this suffice to suffice to determine $H^*(\mathbf{B}PU_n, \mathbb{Z})$, up to degree 10.

Application: The (topological) period-index problem

For a d-dimensional irreducible smooth variety X over an algebraically closed field k, consider α an element of $\mathrm{Br}(X)$, the Brauer group of X, which is a torsion group. Define the period of α, $\mathrm{per}(\alpha)$, to be the order of α in the group $\mathrm{Br}(X)$. On the other hand, α is represented by an Azumaya algebra A over X, which as a free module sheaf over X, has dimension n^2 for some integer n. Let $\mathrm{ind}(\alpha)$ be the greatest common divisor of all n such that α is represented by some Azumaya algebra A of dimension n^2. For a field k we write $\mathrm{Br}(k)$ for $\mathrm{Br}(\mathrm{Spec}\ k)$.
It is well known that $\mathrm{per}(\alpha)|\mathrm{ind}(\alpha)$ and that $\mathrm{per}(\alpha)$ and $\mathrm{ind}(\alpha)$ share the same set of common divisors. Therefore $\mathrm{ind}(\alpha)$ divides some power of $\mathrm{per}(\alpha)$. Based on this observation, we have the following:

Conjecture (Period-Index Conjecture for Function fields)

Let X be an irreducible variety of dimension d over an algebraically closed field k. For any $\alpha \in \mathrm{Br}(X)$, $\mathrm{ind}(\alpha)|\mathrm{per}(\alpha)^{d-1}$.

Application: The (topological) period-index problem

The conjecture has been proved in the case of $d = 2$ by de Jong, but very little is known in the case of higher dimensions. Furthermore, the bound is sharp if it holds at all. See [3] for examples in which $\mathrm{ind}(\alpha) = \mathrm{per}(\alpha)^{d-1}$. The above story has a topological analogy. Let $\mathrm{Br}_{\mathrm{top}}(X)$ be the topological Brauer group of a topological space X. For definitions see, for example, [1]. A theorem of Serre asserts that for a finite CW complex X, $\mathrm{Br}_{\mathrm{top}}(X) \cong \mathrm{Br}'_{\mathrm{top}}(X)$, the latter being the cohomological topological Brauer group defined by $\mathrm{Br}'_{\mathrm{top}}(X) = H^3(X;\mathbb{Z})_{\mathrm{tor}}$, the subgroup of torsion elements of $H^3(X;\mathbb{Z})$.

The conjecture fails in the topological case:

Theorem (Theorem B, [1])

Let n be a positive integer, and let $\epsilon(n)$ denote $ngcd(2, n)$. There exists a connected finite CW complex X of dimension 6 equipped with a class $\alpha \in Br_{top}(X)$ for which $per_{top}(\alpha) = n$ and $ind_{top}(\alpha) = \epsilon(n)n$.

We are interested in seeking the sharp upper bound of $\mathrm{ind}_{\mathrm{top}}(\alpha)$ in terms of $\mathrm{per}_{\mathrm{top}}(\alpha)$, which requires information on the cohomology of $\mathbf{B}PU_n$.

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