MCS 401 Spring 2020
Homework 1

February 4, 2020

1. Problem 1 [Exercise 3-1-4, page 53]:

   (a) Since $0 \leq 2^{n+1} \leq 2 \cdot 2^n$, we have $2^{n+1} = O(2^n)$.

   (b) Since

   $$\lim_{n\to\infty} \frac{2^{2n}}{2^n} = \lim_{n\to\infty} 2^n = \infty$$

   we have that there is no such constant $C > 0$ such that $2^{2n} \leq C(2^n)$. Hence $2^{2n} \neq O(2^n)$. In fact, we have $2^n = o(2^{2n})$, or equivalently $2^{2n} = \omega(2^n)$.

2. Problem 2 [Problem 3-3-a, page 61]:

$2^{2^{n+1}}$	2^{2^n}	$(n+1)!$	$n!$	e^n
$n2^n$	2^n	$(3/2)^n$	$(\lg n)^{\lg n} = n^{\lg\lg n}$	$(\lg n)!$
n^3	$n^2 = 4^{\lg n}$	$n \lg n = \Theta(\lg n!)$	$n = 2^{\lg n}$	$(\sqrt{2})^{\lg n} = \sqrt{n}$
$2^{\sqrt{2\lg n}}$	$\lg^2 n$	$\ln n$	$\sqrt{\lg n}$	$\ln\ln n$
$2^{\lg^* n}$	$\lg^* n$	$\lg\lg^* n$	$n^{1/\lg n}(=2)$	1

   Below are the comparisons that are not so trivial:

   (a) $2^{2^n} = \omega((n+1)!)$: We have

   $$\begin{aligned} \lg(n+1)! &= \sum_{i=1}^{n+1} \lg(i) \approx \frac{1}{\ln 2}\int_1^{n+1} \ln(x)\mathrm{d}x \\ &= \frac{1}{\ln 2}(x\ln x - x)|_1^{n+1} \\ &= \Theta(n \ln n) = o(2^n), \end{aligned}$$

   Hence $(n+1)! = o(2^{2^n})$.

   (b) $n! = \omega(e^n)$: Since $\lg n! = \Theta(n \ln n)$, and $\lg e^n = n \lg e$, and $n \lg e = o(n \ln n)$, we have $e^n = o(n!)$.

   (c) $e^n = \omega(n2^n)$: Since

   $$\begin{aligned} \lim_{x\to\infty} \frac{e^x}{x2^x} &= \lim_{x\to\infty} \frac{e^x}{2^x + x2^x \ln 2} \\ &= \lim_{x\to\infty} \frac{(\frac{e}{2})^x}{1 + x\ln 2} \quad \text{L'hospital} \\ &= \lim_{x\to\infty} \frac{\frac{e}{2}^x(1-\ln 2)}{\ln 2} = \infty, \end{aligned}$$

we have

$$\lim_{n\to\infty} \frac{e^n}{n2^n} = \infty.$$

(d) $(\lg n)^{\lg n} = \Theta(n^{\lg\lg n})$: We have

$$\lg{(\lg n)^{\lg n}} = \lg n \lg\lg n = \lg\lg n \lg n = \lg n^{\lg\lg n}.$$

(e) $\lg n^{\lg n} = \omega(\lg n)!$: Since $m^m = \omega(m!)$, by substituting m with $\lg n$, we get the expression.

(f) $(\lg n)! = \omega(n^3)$: By a), we have $\lg(\lg n)! = \Theta(\lg n \lg\lg n) = \omega(3\lg n) = \lg n^3$ and hence the desired expression.

(g) $2^{\sqrt{2\lg n}} = \omega(\lg n)$: It is easier to see that $2^k = \omega(\frac{k^2}{4})$, and the desired expression is achieved by substituting k with $\sqrt{2\lg n}$.

(h) $\ln\ln n = \omega(2^{\lg^*(n)})$: We have $\lg(\ln\ln n) = \Theta(\lg\lg\lg n)$ Assume that $n = 2^{2^{\cdots^{2^a}}}$, where the number of 2's is k and $0 < a \leq 1$, then we have $\lg^* n = k$ but $\lg(\ln\ln n) = \Theta(\lg\lg\lg n) = \Theta(2^{2^{\cdots^{2^a}}})$, where the number of 2's equals $k-3$.

3. Problem 3 [Exercise 3-4-b, page 62]

We disprove the statement by providing a counterexample. Let us consider

$$f(n) := n^2$$
$$g(n) := n.$$

Then

$$f(n) + g(n) = n^2 + n.$$

However,

$$\Theta(\min(f(n), g(n))) = \Theta(\min(n^2, n)) = \Theta(n).$$

Since

$$\lim_{n\to\infty} \frac{n^2+n}{n} = \infty$$

we obtain that

$$n^2 + n \neq \Theta(n)$$

and conclude the proof.

4. Problem 4 [Exercise 4-5-1, page 96]

For all the four recursive functions, we have $a = 2, b = 4$, and hence $\log_b a = \frac{1}{2}$.

(a) $T(n) = 2T(n/4) + 1$: Since

$$f(n) = 1 = n^0 = O(n^{\frac{1}{2}-\frac{1}{2}}),$$

by the master theorem case 1, we have $T(n) = \Theta(n^{\frac{1}{2}})$.

(b) $T(n) = 2T(n/4) + \sqrt{n}$: Since

$$f(n) = \Theta(n^{\frac{1}{2}}),$$

by the master theorem case 2, we have $T(n) = \Theta(n^{\frac{1}{2}}\log n)$.

(c) $T(n) = 2T(n/4) + n$: Since

$$f(n) = n^1 = \Omega(n^{\frac{1}{2}+\frac{1}{2}}),$$

and

$$2f(n/4) = n/2 \leq 0.9 \cdot n, \forall n \geq 0,$$

by the master theorem case 3, we have $T(n) = \Theta(n)$.

(d) $T(n) = 2T(n/4) + n^2$: Since

$$f(n) = n^2 = \Omega(n^{\frac{1}{2}+\frac{3}{2}}),$$

and

$$2f(n/4) = 1/8 \cdot n^2 \leq 0.5 \cdot n^2, \forall n \geq 0,$$

by the master theorem case 3, we have $T(n) = \Theta(n^2)$.