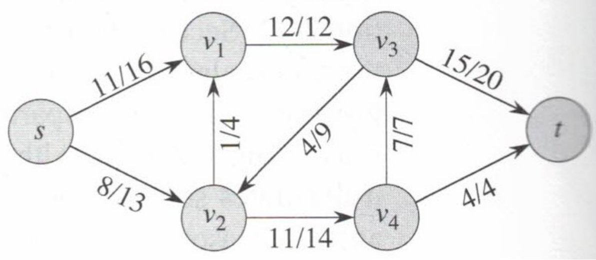
Computer Algorithms I

Spring 2020

Flow networks



$G = (V, E)$ directed graph, source s, sink t

capacity of edges: $c(u, v) \geq 0$

flow along edges: $0 \leq f(u, v) \leq c(u, v)$

Maximum flow problem

flow conservation constraint: if $u \neq s, t$ then

$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v)$$

flow value:

$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s)$$

flow in example has value 19

given a network with capacities, find a flow of maximal value!

Algorithm idea

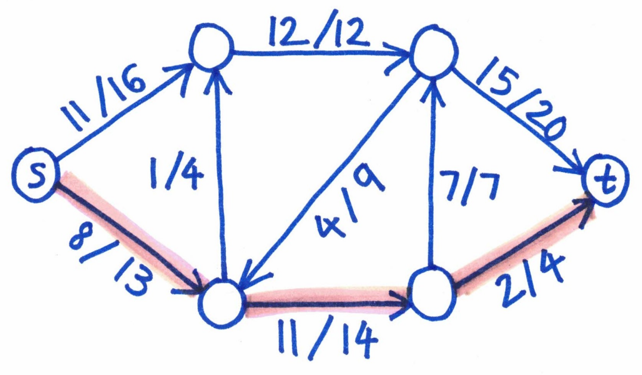
improving a flow: given a flow, modify it to get a flow of larger value!

what if no further improvement is possible?

show that then the flow is optimal!

How to improve a flow?

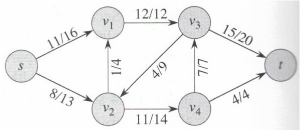
try send more flow along the edges!



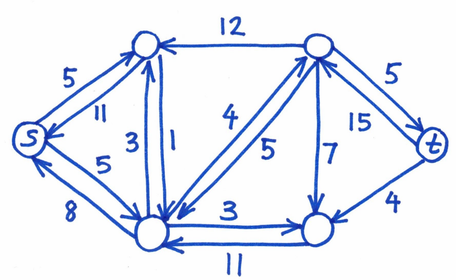
the flow can be increased by 2 units along the path

no such improvement possible in the previous case, and flow is not optimal!

Residual network

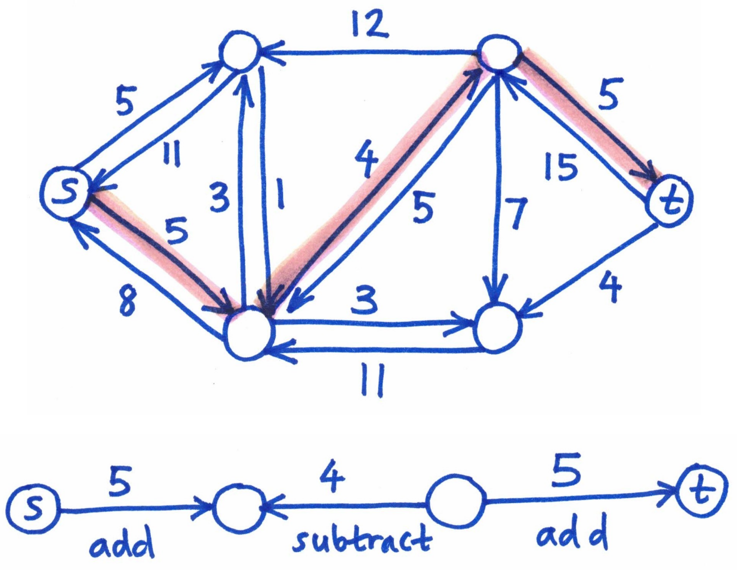


forward / backward edge

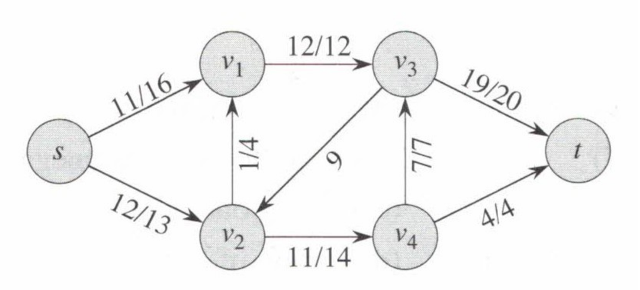


Augmenting path

augmenting path: $s - t$ path in residual network



Improved flow



new flow has value 23

in the residual network no augmenting path

so flow is optimal (?)

Ford-Fulkerson algorithm

```
FORD-FULKERSON(G, s, t)
for each edge (u, v) ∈ G.E
    (u, v).f = 0
while there exists a path p from s to t in the residual network G_f
    c_f(p) = min{c_f(u, v) : (u, v) is in p}
    for each edge (u, v) in p
        if (u, v) ∈ E
            (u, v).f = (u, v).f + c_f(p)
        else (v, u).f = (v, u).f − c_f(p)
```

Definitions (simplified)

residual capacity of edges and reversed edges

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & (u,v) \in E \\ f(v,u) & (v,u) \in E \end{cases}$$

residual network: edges and reverse edges with $c_f > 0$

augmenting path: $s \to t$ path p in residual network

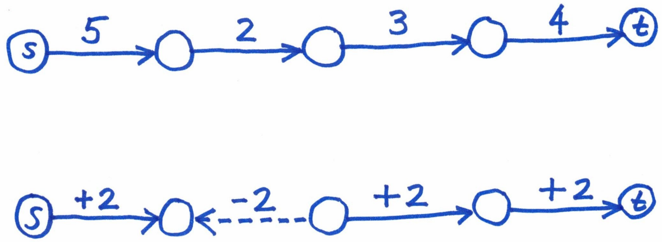
residual capacity of path p: $c_f(p) = \min\{c_f(u,v) : (u,v) \in p\}$

augmented flow:

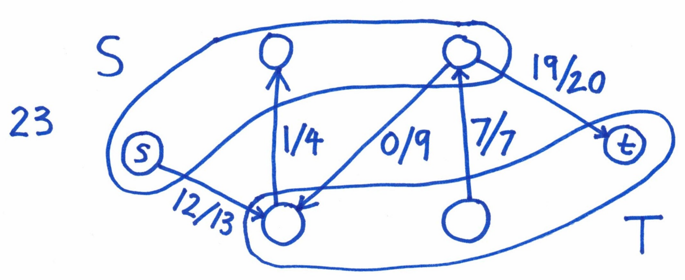
$$(f \uparrow p)(u,v) = \begin{cases} f(u,v) + c_f(p) & (u,v) \in p \\ f(u,v) - c_f(p) & (v,u) \in p \end{cases}$$

value of augmented flow $=$ value of flow $+c_f(p)$

After augmentation we get a flow



Cuts



net flow across cut: $f(S,T) = (12 + 0 + 19) - (1 + 7) = 23$

capacity of cut: $c(S,T) = 13 + 9 + 20 = 42$

$f(S,T) \leq c(S,T)$ for every cut (S,T)

Max-flow min-cut theorem

Theorem

TFAE

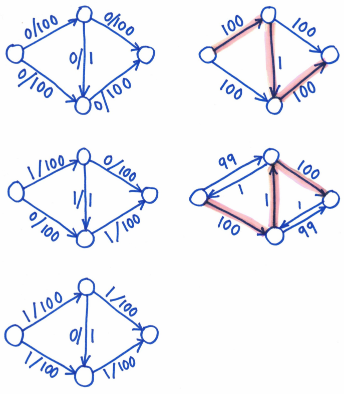
1. *f is a maximum flow*
2. *there is no augmenting path in the residual network for f*
3. *$|f| = c(S, T)$ for some cut (S, T)*

$1 \Rightarrow 2$: any augmenting path could be used to improve $|f|$

$2 \Rightarrow 3$: let S be the vertices reachable from s in the residual network for f, and T be the rest

$3 \Rightarrow 1$: no flow can have value $> c(S, T)$

An example



no running time bound in terms of $|V|$

for irrational capacities may not even terminate!

for integer capacities terminates: every augmentation is at least 1 (also for rational)

Edmonds - Karp algorithm

augment along shortest augmenting path

running time: $O(|V||E|^2)$

there are more efficient algorithms: $O(|V|^3$