FinM 331/Stat 339 Financial Data Analysis,
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Lecture 7
6:30-9:30 pm, 15 February 2010, Ryerson 251 in Chicago
7:30-10:30 pm, 15 February 2010 at UBS in Stamford
7:30-10:30 am, 16 February 2010 at Spring in Singapore

* Gong Xi Fa Cai!
Unfortunately conflicts with Second Day of the Chinese New Year holiday in Singapore, but this is an international program subject to many holidays. Sorry.
7. Method of Moments, Options, Calibration, Implied Volatility, and NonParametric Regression:

7.1. Method of Moments:

The method of moments estimation (MME) of parameters \( \vec{p} = [p_i]_{1 \times m} \) of a given distribution \( F_X(x; \vec{p}) \) of the RV \( X \) by matching sample moments to true moments of \( F_X(x; \vec{p}) \). If there are \( m \) parameters, then a least \( m \) moments are ideally required,

\[
\mu_X^{(k)}(\vec{p}) = \mathbb{E}[X^k] \quad \text{for} \quad k = 1 : m,
\]

(7.1)

where, for example, \( \mu_X^{(1)}(\vec{p}) = \mu \) the usual distribution mean.

Let \( \vec{X} = [X_i]_{1 \times n} \) be a sample of IID RV observations from the assumed known distribution, then the estimated sample means are

\[
\hat{\mu}_n^{(k)} = \frac{1}{n} \sum_{i=1}^{n} X_i^k \quad \text{for} \quad k = 1 : m,
\]

(7.2)

where the sample mean \( \hat{\mu}_n = \hat{\mu}_n^{(1)} \) and the biased sample variance is

\[
S_n^2 = \hat{\mu}_n^{(2)} - \hat{\mu}_n^2.
\]
The estimates of the parameters $\hat{p}_n \approx \vec{p}$ are determined by the moment method equations, assuming a solution exists for $m$ equations in $m$ unknowns,

$$\mu_X^{(i)}(\hat{p}_n) = \hat{\mu}_n^{(i)} \text{ for } i = 1 : m$$  \hspace{1cm} (7.3)

- **7.1.1. Poisson Moment Method Example:**

Suppose the population distribution is a static Poisson so that the RV $X \overset{\text{dist}}{=} P(\Lambda)$ with single parameter $\Lambda = E(p)[X]$. Let the $\vec{X} = [X_i]_{1 \times n}$ be the Poisson IID (IIPD) sample, then estimated MM Poisson parameter is

$$\hat{\Lambda}_n = \bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i,$$  \hspace{1cm} (7.4)

The estimated parameter $\hat{\Lambda}_n$ is an RV that is Poisson distributed, since

$$\text{Prob}[\hat{\Lambda}_n = K] = \text{Prob}\left[\sum_{i=1}^{n} X_i = nK\right] = e^{-n\Lambda} \frac{(n\Lambda)^n}{(nK)!},$$  \hspace{1cm} (7.5)

as will be shown. Knowing the estimate distribution is needed to estimate the error of the estimate.
This can shown directly by first considering the $n = 2$ as a discrete convolution,

$$
\text{Prob}[X_1 + X_2 = 2K] \overset{\text{iid}}{=} e^{-2\Lambda} \sum_{k_1=0}^{\infty} \frac{\Lambda^{k_1}}{k_1!} \sum_{k_2=0}^{\infty} \frac{\Lambda^{k_2}}{k_2!} \mathcal{I}\{k_1 + k_2 = 2K\}
$$

$$
\overset{k_1 \leq 2K}{=} e^{-2\Lambda} \sum_{k_1=0}^{2K} \frac{\Lambda^{k_1} \Lambda^{2K-k_1}}{k_1!(2K)!}
$$

$$
= e^{-2\Lambda} \frac{\Lambda^{2K}}{(2K)!} \sum_{k_1=0}^{2K} \binom{2K}{k_1}
$$

$$
\overset{\text{bin}}{=} e^{-2\Lambda} \frac{\Lambda^{2K}}{(2K)!} (1 + 1)^{2K}
$$

$$
= e^{-2\Lambda} \frac{(2\Lambda)^{2K}}{(2K)!},
$$

so the rest for general $n$ follows by induction though binomial additivity.
It is nice to be able to derive something through first principles as on the prior page, but there is a simpler way using the **moment generating function** (MGF) which has the advantage of easily decomposing up the IID RV parts. Since for a single RV when \( n = 1 \), the exponential transform MGF for the Poisson distribution yields

\[
\text{MGF}^{(p)}[X_i] = \mathbb{E}^{(p)}[e^{tX_i}] = e^{-\Lambda} \sum_{k=0}^{\infty} \frac{\Lambda^k e^{tk}}{k!} = e^{\Lambda(e^t - 1)}, \tag{7.7}
\]

an exponential of an exponential. Hence of the IID sum MGF is

\[
\text{MGF}^{(p)}[\sum_{i=1}^{n} X_i] = \mathbb{E}^{(p)}[\exp(t \sum_{i=1}^{n} X_i)] \overset{loe}{=} \prod_{i=1}^{n} \mathbb{E}^{(p)}[e^{tX_i}]
\]

\[
= \prod_{i=1}^{n} e^{\Lambda(e^t - 1)} \overset{loe}{=} e^{n\Lambda(e^t - 1)}, \tag{7.8}
\]

which is the same form as \((7.7)\), except with the parameter \( \Lambda \) replaced by \( n\Lambda \) and thus proving \((7.5)\), so \( \mathbb{E}^{(p)}[\sum_{i=1}^{n} X_i] = n\Lambda \). Consequently, we have that \( \hat{\Lambda}_n \) is **unbiased**, i.e.,

\[
\mathbb{E}^{(p)}[\hat{\Lambda}_n] = \frac{1}{n} \mathbb{E}^{(p)}\left[ \sum_{i=1}^{n} X_i \right] = \frac{1}{n} n\Lambda = \Lambda. \tag{7.9}
\]
Also, the estimated parameter sample variance is

$$\text{Var}^{(p)}[\hat{\Lambda}_n] = \frac{1}{n^2} \text{Var}^{(p)}\left[\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2} n\Lambda = \frac{\Lambda}{n} = \text{SE}^2[\hat{\Lambda}_n]$$

(7.10)

and the standard error is

$$\text{SE}[\hat{\Lambda}_n] = \sqrt{\text{Var}^{(p)}[\hat{\Lambda}_n]} = \frac{\Lambda}{\sqrt{n}},$$

(7.11)

which goes to zero as $n \to \infty$. However, $\text{SE}[\hat{\Lambda}_n]$ can be used as an estimate of the order of the error in $\hat{\Lambda}_n$ and using the estimate for the standard error gives an actual estimate of the standard error, i.e.,

$$\text{SE}[\hat{\Lambda}_n] \simeq \frac{\hat{\Lambda}_n}{\sqrt{n}}.$$  

(7.12)

As $n \to \infty$, the $\hat{\Lambda}_n$ can be asymptotically approximated by a normal distribution using the central limit theorem,

$$\hat{\Lambda}_n \xrightarrow{\text{dist}} \mathcal{N}(\Lambda, \Lambda/\sqrt{n})$$

(7.13)

and can be approximated by the 2-sigma or 95% confidence interval rule,

$$\hat{\Lambda}_n \simeq \Lambda(1 \pm 2\hat{\Lambda}_n/\sqrt{n}).$$

(7.14)
7.1.2. Normal Moment Method Example:

Now suppose we have a normal distribution \( \mathcal{N}(\mu, \sigma) \) and an sample of \( n \) IID normal (IIND) observations \( \vec{X} \). The moment method involves two parameters \( \vec{p} = (\mu, \sigma^2) \) and two true moments, the first moment

\[
\mu_X^{(1)}(\vec{p}) = \mathbb{E}^{(n)}[X] = \mu
\] (7.15)

and the second moment

\[
\mu_X^{(2)}(\vec{p}) = \mathbb{E}^{(n)}[X^2] = \sigma^2 + \mu^2.
\] (7.16)

As before the **mean estimate** is

\[
\hat{\mu}_n \equiv \bar{X}_n,
\] (7.17)

but now the **variance estimate** is

\[
\hat{\sigma}^2_n \equiv (\bar{X}^2)_n - \bar{X}^2_n.
\] (7.18)

The **mean estimate is unbiased** since

\[
\mathbb{E}^{(n)}[\bar{X}_n] = \mathbb{E}^{(n)}[\hat{\mu}_n] = \frac{1}{n} \mathbb{E}^{(n)}[X_i] = \mu.
\] (7.19)
The variance of the mean estimate is then

\[ \text{Var}^{(n)}[\hat{\mu}_n] = \mathbb{E}^{(n)}[(\bar{X}_n - \mu)^2] \]

\[ = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E}^{(n)}[(X_i - \mu)(X_j - \mu)] \]

\[ = \frac{1}{n^2} \sum_{i=1}^{n} \sigma^2 = \frac{\sigma^2}{n} = \text{SE}^2[\hat{\mu}_n] \quad (7.20) \]

and the standard error of the mean estimate is

\[ \text{SE}[\hat{\mu}_n] = \frac{\sigma}{\sqrt{n}}. \quad (7.21) \]

It can also be shown that

\[ \hat{\mu}_n = \bar{X}_n \overset{\text{dist}}{=} \mathcal{N}(\mu, \sigma/\sqrt{n}). \quad (7.22) \]

so normally distributed for any \( n \), not just asymptotically normal distributed.
For the normal distribution $F_{X_i}^{(n)}(x; \mu, \sigma^2)$, the MGF for a single IID RV is

$$\text{MGF}^{(n)}[X_i] = E^{(n)}[e^{tX_i}] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} dx \ e^{-\frac{(x-\mu)^2}{2\sigma^2} + tx}$$

(7.23)

so the for the sample mean,

$$\text{MGF}^{(n)}[\bar{X}_n] = E^{(n)}[e^{t\sum_{i=1}^{n} X_i/n}] \overset{\text{ind}}{=} \prod_{i=1}^{n} E^{(n)}[e^{tX_i/n}]$$

(7.24)

$$= \prod_{i=1}^{n} e^{\mu t/n + \sigma^2(t/n)^2/2} = e^{\mu t + (\sigma^2/n)t^2/2},$$

justifying the estimated mean distribution in (7.22).

Further, Rice (2007; p. 263 with cross-references) shows that the estimated variance scaled by $SE^2[\hat{\mu}_n]$ behaves as a Chi-squared distribution with $n - 1$ degrees of freedom, i.e.,

$$\frac{\hat{\sigma}^2_n}{\sigma^2/n} \overset{\text{dist}}{\sim} \chi^2_{n-1}.$$  

(7.25)
7.1.3. Gamma Moment Method Example:
The gamma distribution is a generalization of the exponential distribution. Its density on \((0, \infty)\) is given by

\[
f^{(g)}_{\lambda, \alpha}(x) = \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)}, \quad (7.26)
\]

where \(\alpha > 0\) and \(\lambda > 0\) for integrability, while its normalization depends on the gamma function \(\Gamma(\alpha)\)

\[
\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx, \quad (7.27)
\]

such that \(\Gamma(1) = 1\) and \(\Gamma(\alpha + 1) = \alpha \Gamma(\alpha) = \alpha!\). The true moments can be deduced from the gamma function MGF, letting \(X\) be a gamma distributed IID RV and \(t < \lambda\), so from the MGF moment coefficients, \(MGF^{(g)}[X] = E^{(g)}[e^{tX}] = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\lambda-t)x} dx\)

\[
= \frac{\lambda^\alpha}{\Gamma(\alpha)(\lambda-t)^\alpha} \int_0^\infty y^{\alpha-1} e^{-y} dy = \frac{1}{(1-t/\lambda)^\alpha} \quad (7.28)
\]

\[
\sim 1 + \frac{\alpha t}{\lambda} + \frac{\alpha(\alpha + 1)t^2}{2\lambda^2} = 1 + \mu^{(1)}_X(\vec{p})t + \mu^{(2)}_X(\vec{p}) \frac{t^2}{2}.
\]
Reading the moments from the expansion in the last line of (7.28), we have the true the first moment,

$$\mu_X^{(1)} = \mathbb{E}(g)[X] = \frac{\alpha}{\lambda}$$  \hspace{1cm} (7.29)

and the second moment

$$\mu_X^{(2)} = \mathbb{E}(g)[X^2] = \frac{\alpha(\alpha + 1)}{\lambda^2}. \hspace{1cm} (7.30)$$

Since the moment method produces moments first and the parameters can be derived second, we need the inverse of the nonlinear relationship between the distribution defined parameters and the moments,

$$\lambda = \frac{\mu_X^{(1)}}{\mu_X^{(2)} - \left(\mu_X^{(1)}\right)^2} \hspace{1cm} (7.31)$$

and

$$\alpha = \lambda \mu_X^{(1)} = \frac{\left(\mu_X^{(1)}\right)^2}{\mu_X^{(2)} - \left(\mu_X^{(1)}\right)^2}. \hspace{1cm} (7.32)$$
Letting $\bar{X} = [X_i]_{1 \times n}$ be a set of IID gamma distributed (IIGD) observations, the parameter estimates are

$$\hat{\lambda}_n = \frac{\hat{\mu}_n}{\hat{\mu}^{(2)}_n - (\hat{\mu}_n)^2} = \frac{\hat{\mu}_n}{\hat{\sigma}_n^2}$$ (7.33)

and

$$\hat{\alpha}_n = \frac{\hat{\mu}_n^2}{\hat{\sigma}_n^2}. \quad (7.34)$$

{Remark: Due to the complexity of estimated parameters, $\hat{\lambda}_n$ and $\hat{\alpha}_n$, in relation to the estimated moments, $\hat{\mu}_n$ and $\hat{\mu}^{(2)}_n$, finding the distribution of the parameter estimates for estimating the parameter errors is too difficult. Rice (2007, pp. 263-266) suggests bootstrapping a large family of samples $\{\bar{X}_j : j = 1 : M\}$ to simulate empirical distributions of $\hat{\lambda}_n$ and $\hat{\alpha}_n$, using that to estimate the errors.}
However, the parameter distribution of the moment estimates can be calculated using the MGF, but the product distribution needs to be generated. Hence, for the sample mean $\hat{\mu}_n = \overline{X}_n$,

$$
\text{MGF}^{(g)}[\overline{X}_n] = E(g) [ e^{t \sum_{i=1}^n X_i / n}]^{\text{ind}} \prod_{i=1}^n E(g) [ e^{t X_i / n}]
$$

$$
= \prod_{i=1}^n \left( \frac{1}{1 - t/(n \lambda)} \right)^{\alpha} = \frac{1}{(1 - t/(n \lambda))^{n\alpha}},
$$

so

$$
\hat{\mu}_n = \overline{X}_n \overset{\text{dist}}{\sim} F^{(g)}_X(x; n\lambda, n\alpha). \tag{7.36}
$$

Thus

$$
E(g)[\overline{X}_n] = \frac{\alpha}{\lambda}, \tag{7.37}
$$

$$
E(g)[\overline{X}_n^2] = \frac{n\alpha(n\alpha + 1)}{(n\lambda)^2} = \frac{\alpha}{\lambda} \left( \frac{\alpha}{\lambda} + \frac{1}{n\lambda} \right) \tag{7.38}
$$

and

$$
\text{Var}^{(g)}[\overline{X}_n] = \frac{\alpha}{\lambda} \left( \frac{\alpha}{\lambda} + \frac{1}{n\lambda} \right) - \left( \frac{\alpha}{\lambda} \right)^2 = \frac{\alpha}{n\lambda^2}. \tag{7.39}
$$
The standard error of the estimated first moment is then,

$$SE[\bar{X}_n] = \frac{1}{\lambda} \sqrt{\frac{\alpha}{n}}. \quad (7.40)$$

7.1.4. Consistent Parameter Estimates:

**Definition of Consistency:** A parameter estimate $\hat{p}_n$ of a single parameter $p$ from sample size $n$ is a consistent estimate if $\hat{p}_n$ converges in probability to $p$ as $n \to \infty$,

$$\text{Prob}[|\hat{p}_n - p| > \varepsilon] \to 0 \quad \text{as} \quad n \to \infty, \quad (7.41)$$

for any $\varepsilon > 0$.

The weak law of large numbers is support for this definition. Consistency in probability justifies the use of the standard error of the estimate,

$$SE[\hat{p}_n] = \sigma/\sqrt{n} \quad \text{where} \quad \sigma = \sigma(p) \quad \text{is the true standard error.}$$

An approximate form of consistency is the use of the estimated standard error such that

$$SE[\hat{p}_n] \approx \hat{SE}[\hat{p}_n] \equiv \sigma(\hat{p}_n)/\sqrt{n}, \quad (7.42)$$

assuming continuity of $\sigma(p)$ then $\hat{p}_n \to p$ implies $\sigma(\hat{p}_n) \to \sigma(p)$, in theory.
MME versus MLE Remarks: The method of moments parameter estimation (MME) is very old and predates the maximum likelihood method of estimating parameters. Maximum likelihood method estimation was introduced by the legendary statistician R. A. Fisher as an improvement over the method of moments to have a higher probability of closer estimations, as the name maximum likelihood suggests, so MLE is said to be more efficient. The method of moments can produce improper answers with small samples, so large samples are important. The same can be said for a larger number of parameter, since it may be difficult to solve the system of moment equations.
7.1.5. Generalized Moment Method:
The are a number of variants of the ordinary moment method and one was introduced by Lars Peter Hansen (1982) of the University of Chicago called the generalized moment method (GMM) which uses a general function of the observation RV $X$ and the parameter $\vec{p}$, with expectation
\[
\bar{\mu}(\vec{p}) = \mathbb{E}_X [\vec{g}(X, \vec{p})],
\] (7.43)
such that the otherwise general $\vec{g}$ has mean zero,
\[
\bar{\mu}(\vec{p}) = \mathbb{E}_X [\vec{g}(X, \vec{p})] = \vec{0},
\] (7.44)
in analogy to the critical point conditions of MLE.

The sample mean is the usual,
\[
\hat{\mu}_n(\vec{p}) = \frac{1}{n} \sum_{i=1}^{n} \vec{g}(X_i, \vec{p})
\] (7.45)
and since by the law of large numbers $\hat{\mu}_n(\vec{p}) \to \bar{\mu}(\vec{p})$ as $n \to \infty$, leading to the parameter estimation condition,
\[
\hat{\mu}_n(\hat{p}_n) \simeq \vec{0}.
\] (7.46)
The optimal objective is model on the idea of robust, **weighted least squares** with the sample size dependent, positive definite weight matrix $W_n$ such that the **optimal estimated parameter vector** is given by

$$\hat{p}_n = \arg\min_{\tilde{p}} \left[ \mu_n^\top(\tilde{p}) W_n \mu_n(\tilde{p}) \right].$$  \hfill (7.47)

A good, efficient choice is the inverse of the $\vec{g}$-covariance matrix,

$$\text{Cov}\left[ \vec{g}(X, \vec{p}) \vec{g}^\top(X, \vec{p}) \right] = E[\vec{g}(X, \vec{p}) \vec{g}^\top(X, \vec{p})]$$

$$\simeq \frac{1}{n} \sum_{i=1}^{n} \vec{g}(X_i, \hat{p}_n^{(0)}) \vec{g}^\top(X_i, \hat{p}_n^{(0)}).$$  \hfill (7.48)

$$= \frac{\vec{g}\vec{g}^\top}{n} \equiv (W_n^{(0)})^{-1},$$

recalling that $\vec{g}$ has mean zero, where $\hat{p}_n^{(0)}$ is a starting or current parameter estimate. The parameter estimate using such a weight matrix estimate can be computed using a robust or nonlinear regression or an optimal search method.$^a$

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PS: Another often used advanced moment method is the **efficient moment method (EMM)** which successively combines the efficiency of the maximum likelihood estimation (MLE) and simulation moment method (SMM). For a compact, readable account and SAS example code see *Efficient Method of Moments Estimation of a Stochastic Volatility Model*. The EMM was used in a paper often quoted in class for results showing the importance of both jumps and stochastic volatility: T. G. Andersen, L. Benzoni and J. Lund, “An Empirical Investigation of Continuous-Time Equity Return Models,” *J. Fin.*, vol. 57, 2002, pp. 1239–1284.
7.2. Vanilla, European Options:

7.2.1. Black-Scholes European Call and Put Options:

With **delta-hedging** to eliminate the risk due to volatility terms and arbitrage-free conditions restricting portfolio growth to the risk-free rate, \( r \), we are effectively dealing with the modified, underlying stock price \( S(t) \) diffusion SDE,

\[
dS(t) = S(t)(rdt + \sigma dW(t)), \quad S(0) = S_0, \tag{7.49}
\]

where both \( r \) and the volatility \( \sigma \) are assumed to be constant, although that is not necessary.

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\[a\] In part, adapted from Carmona (’04) Ch. 4.; Hull (6th Ed., ’06); D. Higham (’04); Hanson’s *Applications in Financial Engineering*, Chapter 10; CBOE’s Stock Options brochure [http://www.cboe.com/LearnCenter/pdf/understanding.pdf](http://www.cboe.com/LearnCenter/pdf/understanding.pdf)

\[b\] The Greek delta, \( \Delta = \frac{\partial C}{\partial S} \) is used to **hedge** away the uncertainty risk, where for example \( C \) is the call option price and \( S \) is the stock price.
Black and Scholes (Spring 1973) derived a solution for a **European style call option** contract to buy the stock for a **strike price** $K$ at a specified contract maturity date paying the **call option contract price** or **premium** $C(S(t), t; K, T, r, \sigma)$ at current time $t$ when underlying price $S(t)$, all governed by the final gain or strike payoff function

$$C(S(T), T; K, T, r, \sigma) = G(S(T), K) = \max[S(T) - K, 0].$$  \hspace{1cm} (7.50)

Presumably, $G(S_0, K) = 0$, i.e., $S_0 < K$, otherwise there would be no incentive to sell the contract to the buyer, the buyer betting that the stock price will rise over $K$ and if $G(S(T), K) = 0$ the rational buyer would walk away from the contract since the stock could be purchased more cheaply in the market.

Note that index options\(^{a}\) are different, mainly that there is a cash settlement replacing the opportunity to buy stock at $K$ and thus is closer to real betting.

\(^{a}\)Again, see CBOE’s Understanding Index Options brochure: [http://www.cboe.com/LearnCenter/pdf/understandingindexoptions.pdf](http://www.cboe.com/LearnCenter/pdf/understandingindexoptions.pdf)
Black and Scholes’ well-known solution formula uses the solution of the SDE (also a backward problem for a PDE) and is

\[ C^{(bs)}(S, t; K, T, r, \sigma) = SF_X^{(n)}(d_1; 0, 1) - e^{-r(T-t)}KF_X^{(n)}(d_2; 0, 1), \]  

(7.51)

where

\[ d_1 = d_1(S/K, T-t, r, \sigma) \equiv (\log(S/K) + (r + \sigma^2/2)(T-t))/(\sigma \sqrt{T-t}) ; \]

\[ d_2 = d_2(S/K, T-t, r, \sigma) = d_1(S/K, T-t, r, \sigma) - \sigma \sqrt{T-t}, \]

(7.52)

noting the natural dependence is on the time to maturity \( \tau \equiv T-t \), also called the time-to-go, and the moneyness, the ratio

\[ M = M(\tau) \equiv S e^{r\tau}/K, \]  

so that \( \log(M) = \log(S/K) + r\tau \) is the log-moneyness.

\[ ^a\text{Moneyness can also be defined as the reciprocal } K/(Se^{r\tau}) \text{ when the focus is on } K \text{ as for a put option. See, for instance, Prof. R. Lee’s (2004) thorough paper on } \text{Implied Volatility}. \]
Note that \( Se^{r(T-t)} \) is the future value of the current price \( S \) to maturity from \( t \) compounded at the risk-free rate \( r \) or alternately \( e^{-r(T-t)}K \) is the present value of the strike price \( K \), available at maturity \( T \), but discounted at the risk-free rate \( r \) back to present time \( t \). If \( M = 1 \) then the option is at the money (ATM), else if \( M > 1 \) then it is in the money (ITM) for a call option, else \( M < 1 \) then it is out the money (OTM) for a call option, but ITM for the put option. Note that at exercise, \( M = S(T)/K \), so then ITM or ATM mean \( K \leq S(T) \). Also, ITM is not the same as in the profit “ITP”, since that requires a net profit or Profit = \( S - K - \text{Premium} \) > 0 for the call option.

\(^{a}\)In general, the discount rate \( \beta \) is different than the interest rate \( r \) as the present value differs from the future value. They both can depend on time \( t \).
Thus, for financial and numerical purposes, we may define a more computational finance form of the call option price
\[
\tilde{C}^{(bs)}(M, \tau; \sigma) \equiv C^{(bs)}(e^{-r\tau}KM, T-\tau; K, T, r, \sigma)/(e^{-r\tau}K) = MF_X^{(n)}(d_1; 0, 1) - F_X^{(n)}(d_2; 0, 1),
\]
(7.53)
where \(d_1 = \log(M)/\tilde{\sigma} + \tilde{\sigma}/2\) and \(\tilde{\sigma} \equiv \sqrt{\sigma^2 \tau}\) which is the scaled volatility. You can verify that the scaled call price goes to the correct limit as \(\tau \to 0^+\).

The corresponding European put option is a contract to sell stock to the contract maker at \(K\) at \(T\) under an asymmetric version of the payoff,
\[
\mathcal{P}(S(T), T; K, T, r, \sigma) = G(-S(T), -K) = \max[K - S(T), 0],
\]
(7.54)
with solution,
\[
\mathcal{P}^{(bs)}(S, t; K, T, r, \sigma) = -SF_X^{(n)}(-d_1; 0, 1) + e^{-r(T-t)}KF_X^{(n)}(-d_2; 0, 1).
\]
(7.55)
Note that moneyness for the put should literally be the opposite of money for the call, i.e. \(1/M\) makes more sense for the put option.
Equations (7.51) and (7.55) connect a maximum and replicated portfolio derivation called the **put-call parity**, 

\[ P^{(bs)}(S, t) + S = C^{(bs)}(S, t) + e^{-r(T-t)} K, \]  

(7.56)

suppressing parameter arguments.

However, as we have previously discussed, the Black-Scholes model, despite its extensive service in quantitative finance for almost 37 years, has many deficiencies, like unrealistic constant coefficients (though Merton’s (also Spring 1973) justification paper generalized it to variable coefficients and many other things), lack of fat tails subsequent poor risk assessment, skewness, jumps, stochastic volatilities, etc.
7.2.2. Market Calibration and Implied Volatility:

One work-around the deficiencies with Black-Scholes formula, is to find a volatility that better fits market values of the instrument of interest, say the European call option. Hence, given market data $C^{(mkt)}(K_i, T_i)$ for a discrete number of strikes $K_i$ and corresponding maturities $T_i$ for any given call option, the financial engineer will make an estimate of the volatility, and possibly other parameters, that is implied by option market rather than the underlying stock market.

When the underlying stock price data is used to estimate the underlying volatility, then the log-return $LR_i \equiv \log(S_{i+1}/S_i)$ is used, with estimated mean

$$\overline{LR} = \frac{1}{n} \sum_{i=1}^{n} LR_i.$$ (7.57)
The unbiased estimated volatility

\[ \hat{\sigma}^{(\text{hist})} = \sqrt{\frac{1}{(n-1)\Delta t} \sum_{i=1}^{n} (LR_i - \overline{LR})^2} \]  

is called the historical volatility; note that in the difference approximation to the asset SDE \((S\Delta E)\), \(E[LR_i] = (\hat{\mu} - \hat{\sigma}^2 / 2) \Delta t\) rather than the risk-neutral \((r - \hat{\sigma}^2 / 2) \Delta t\).

However, the call market prices are not usually given directly, but, for instance in the delayed quotes at the Chicago Board of Options Exchange (CBOE)\[^a\], they are given in terms of the latest bid and ask quotes, so usually one takes the average of the bid and the ask quotes for \(C^{(\text{mkt})}(K_i, T_i)\) for each contract pair \((K_i, T_i)\).

\[^a\]CBOE Delayed Market Quotes page is found for download at the URL:

http://www.cboe.com/delayedquote/QuoteTableDownload.aspx
The option market implied estimate is the so-called **Black-Scholes implied volatility (IV)**, $\sigma_{i}^{(iv)}$, by solving the inverse problem, matching the BS call price to the market call price, a\[ C^{(bs)}(M_i, T_i; \sigma_{i}^{(iv)}) = C^{(mkt)}(K_i, T_i), \] defining $\sigma_{i}^{(iv)}$ for each $i = 1:n$, given options data \{ $K_i, T_i, S_0$ \}, where $M_i = S_0/K_i$, $C^{(bs)}$ is given in (7.51), and for fixed $r$ and $t = 0$ as the current time.

One problem in estimating volatility or variance is that they can not be directly observed but must be deduced from other observations like stock or option prices. There are also many methods for estimating implied volatility including Newton’s method, maximum likelihood, kernel smoothing, and Monte Carlo, but for a single scalar variable like $\sigma$ the derivative-free root-finder **fzero** of **MATLAB** could be used.

---

\(^{a}\)Note that \(\text{vega} = \nu = \partial C^{(bs)}/\partial \sigma > 0\), a volatility sensitivity measure. Hence, the inverse should exist for Black-Scholes. See D. Highham (2004), *An Introduction to Financial Option Valuation*, p. 101 and 132; also for a simple justification of put-call parity.
However, there is not that much strike-maturity data, so **pooled data is sometimes used**, e.g., short maturity, medium maturity and long maturity options, or long-run historical data. Getting historical data has been harder to get, e.g., for European options, in the public domain, unless available in a company or business school.
7.2.3. Risk-Neutral Option Pricing and Implied Volatility:
While a relatively simple solution to European call or put option pricing problem with delta ($\Delta^{(bs)} \equiv \partial C^{(bs)}/\partial S$) hedging, the multiple sources of randomness in jump-diffusions or stochastic-volatility jump-diffusions do not allow for delta hedging to eliminate the purely diffusive risks. However, a risk-neutral formulation of the discounted, expected, conditional payoff simulates the principal properties of delta hedging. In addition, the arbitrage-free condition must be used by setting the instantaneous mean rate to the risk-free rate,

$$E[dS(t)|S(t) = s]/(sdt) = r,$$

(7.60)
e.g., $\mu = r$ for linear diffusions or $\mu + \lambda \bar{v} = r$ for linear compound-jump-diffusions.
Thus, for linear diffusions and more general cases, the current risk-neutral (RN) European style call or put option prices are given by

\[
\begin{align*}
\left[ C, P \right]^{(rn)}(s, t; K, T, r, \bar{\theta}) &= e^{-r(T-t)} \\
\cdot E^{(rn)}[G(\pm S(T), \pm K)|S(t) = s],
\end{align*}
\] (7.61)

where again \( G(\pm S, \pm K) = \max[\pm(S - K), 0] \equiv [\pm(S - K)]_+ \) is the payoff function and \( \bar{\theta} \) is the vector of other model parameters.
7.2.4. **Risk-Neutral Option Pricing Application to Compound-Jump-Diffusion (CJD) Underlying Asset:**

As a good example of a genuine risk-neutral options model, consider the risk-neutral version of the constant-coefficient, zero-one jumps on \((t, t + \Delta t]\) compound-Poisson, jump-diffusion (CJD) SDE asset price model (Merton, 1976) underlying the option,

\[ dS^{(rn)}(t) = S(t)((r - \lambda \bar{\nu})dt + \sigma dW(t)) + \nu(Q)dP(t; Q), \quad (7.62) \]

where the IID \(\nu(Q) = \exp(Q) - 1\), \(\bar{\nu} = \mathbb{E}_Q[\nu(Q)]\) and the required risk-neutral property is \(\mathbb{E}[dS(t)|S(t)] = rS(t)dt\). Converting to the log-return variable \(Y(t) = \log(S(t))\) using the hybrid independent continuous and jump process stochastic chain, leads to a state independent right-hand-side,

\[ dY^{(rn)}(t) = (r - \sigma^2/2 - \lambda \bar{\nu})dt + \sigma dW(t) + QdP(t; Q), \quad (7.63) \]

where \(Q = \log(1 + \nu(Q))\) has been used, provided \(\nu(Q) > -1\).
Integrating from current time $t$ to final, contract exercise time $T$, with $	au = T - t$, and exponentiating yields,

$$S^{(rn)}(T) = S(t)\exp\left((r - \sigma^2/2 - \lambda\nu)\tau + \sigma W(\tau) + \sum_{j=1}^{P(\tau)} Q_j\right).$$  \hspace{1em} (7.64)

Since the time-to-maturity time-interval $(t, T]$ is not infinitesimal nor is sort of small a the trading day in years, we have to count the number of jumps in $(t, T]$ and the probable number of jumps in $(t, T]$ is the same as the number of jumps in $(0, T-t] = (0, \tau]$, so $P(T) - P(t) = P(\tau)$ and

$$\int_t^T QdP(t) = \sum_{j=1}^{P(\tau)} Q_j I\{P(\tau) > 0\} = \sum_{j=1}^{P(\tau)} Q_j,$$  \hspace{1em} (7.65)

with the no-jump convention that $\sum_{j=1}^{0} Q_j \equiv 0$ and $I\{S\}$ is the indicator function for set $S$. 

\begin{align*}
\text{FINM 331/Stat 339 W10 Financial Data Analysis} & \quad \text{Lecture7-page32} \quad \text{Floyd B. Hanson}
\end{align*}
The property of both diffusion and time-homogeneous Poisson processes that their increments depend only on the time step is called the **stationary property**, then \( W(T') - W(t) = W(\tau) \) and \( P(T') - P(t) = P(\tau) \).

Next the scaled diffusion form, \( W(\tau) = \sqrt{\tau}Z \) where \( Z \) is a mean-zero, variance-one normal RV. For notational simplicity, the \( P(\tau) = k \) jump-sum is \( S_k \equiv \sum_{j=1}^{k} Q_j \) of jump-amplitudes.

Substituting formula (7.64) for \( S(T) \) into the risk-neutral formula for the call option price with parameter vector \( \vec{p} = [K, T, r, \vec{\theta}] \), along with normal density, Poisson distribution and IID RV expectation, yields,

\[
C^{(rn)}(s, t; \vec{p}) = e^{-r\tau} \sum_{k=0}^{\infty} p_k(\lambda \tau) \mathbb{E}_{S_k} \left[ \int_{-\infty}^{\infty} dz f_Z^{(n)}(z; 0, 1) \cdot \max \left[ s e^{(r - \sigma^2/2 - \lambda \nu)\tau + \sigma \sqrt{\tau}z + S_k - K, 0} \right] \right].
\]

(7.66)
Next, the payoff maximum operator will be eliminated by finding the break even point (BEP) for the payoff by finding the zero $Z_0$ of the first argument

$$s e (r - \sigma^2/2 - \lambda \nu) \tau + \sigma \sqrt{\tau} Z_0 + S_k - K = 0,$$

whose solution is

$$Z_0^{\text{alg}} = - \left( \log(s/K) + (r - \sigma^2/2 - \lambda \nu) \tau + S_k \right) / \sqrt{\sigma^2 \tau}$$

$$- d_2 + (\lambda \nu \tau + S_k) / \sqrt{\sigma^2 \tau}$$

$$\equiv -d_{2,k} \equiv -d_{1,k} + \sigma \sqrt{\tau},$$

borrowing from Black and Scholes normal argument notation, since we are following the risk-neutral procedure for the Black-Scholes formula modified for jumps.
Substitution back into the current version of risk-neutral call option price, cutting off the left tail of the integral,

\[
C^{(rn)}(s, t; \bar{p}) = e^{-r\tau} \sum_{k=0}^{\infty} p_k(\lambda \tau) \mathbb{E}_{S_k} \left[ \int_{-d_{2,k}}^{\infty} dz f_Z^{(n)}(z; 0, 1) \right] \\
\left. \left( s e^{(r - \sigma^2/2 - \lambda \nu) \tau + \sigma \sqrt{\tau} z + S_k - K} \right) \right]^{\text{alg}} \sum_{k=0}^{\infty} p_k(\lambda \tau) \mathbb{E}_{S_k} \left[ \\
se^{-\sigma^2/2 + \lambda \nu) \tau + S_k} \int_{-d_{2,k}}^{\infty} dz f_Z^{(n)}(z; 0, 1) e^{\sigma \sqrt{\tau} z} \\
- e^{-r\tau} K \int_{-d_{2,k}}^{\infty} dz f_Z^{(n)}(z; 0, 1) \right].
\]

(7.69)
Either using the complete the square technique or letting \( y = z - \sigma \sqrt{\tau} \), along with \( \exp(\sigma Z) \) normal integral formula, then

\[
C^{(rn)}(s, t; \vec{p}) = \sum_{k=0}^{\infty} p_k(\lambda \tau) \mathbb{E}_{S_k}\left[ s e^{S_k - \lambda \nu \tau} F_Z^{(n)}(d_{1,k}; 0, 1) - e^{-r \tau} K F_Z^{(n)}(d_{2,k}; 0, 1) \right],
\]

where we have used the identities, \(-d_{2,k} \equiv -d_{1,k} + \sigma \sqrt{\tau}\) and \(\int_{-\infty}^{\infty} = \int_{-d}^{d}\) for even integrable integrands. Finally, we form the compound-jump-diffusion risk-neutral European call option price as a compound-Poisson mixture of Black-Scholes call option prices,

\[
C^{(rn)}(s, t; \vec{p}) = \sum_{k=0}^{\infty} p_k(\lambda \tau) \cdot \mathbb{E}_{S_k}\left[ C^{(bs)}(s e^{S_k - \lambda \nu \tau}, t; K, t + \tau, r, \sigma) \right],
\]

where, for example, in the case of a single uniform jump amplitude model, \( \vec{\theta} = [\sigma, \lambda, a, b]^\top \). The formula reduces to the Black-Scholes if \( \lambda = 0 \) and \( \vec{\theta} = [\sigma] \).
Due to the complex nature of this call option price formula with Poisson and IID RV expectations, perhaps Monte Carlo simulations would be most practical, especially since a Poisson simulation of the number of jumps would keep the Poisson sum finite and the Poisson sum and the IID RV expectation could be combined. For instance, following Zhu and Hanson (2005), we can replace the Poisson sum and the sum $S_k$ in the single uniform distribution case, with sample IID Poisson variates $P_i$ for $i = 1:n$ and standard RVs $U_{i,j}$ for $j = 1:P_i$, on $(a,b)$ by the estimate

$$\hat{S}_i = \sum_{j=1}^{P_i} Q_{i,j} = \sum_{j=1}^{P_i} (a + (b-a)U_{i,j}) = aP_i + (b-a)\sum_{j=1}^{P_i} U_{i,j}. \quad (7.72)$$

---

\textsuperscript{a}Zhu and Hanson (2005) give elaborate Monte Carlo procedures with variance reduction techniques for European options in a jump-diffusion model with uniformly distributed jump-amplitudes, showing that jump-diffusion options are worth more than Black-Scholes diffusion options, in the paper at

Then for the Poisson sample size $n$, the Monte Carlo estimate of the call option price, starting at $t = 0$, is simply,

$$
\hat{C}_n = \frac{1}{n} \sum_{i=1}^{n} C^{(bs)}(S_0 e^{\hat{S}_i - \lambda \nu T}, 0; K, T, r, \sigma) \equiv \frac{1}{n} \sum_{i=1}^{n} C_i^{(bs)}, \quad (7.73)
$$

where $C_i^{(bs)}$ is IID compound Poisson variate along with $\hat{S}_i$, so

$$
\hat{C}_n \rightarrow C^{(rn)}(S_0, 0; K, T, r, \sigma) \text{ as } n \rightarrow \infty \quad (7.74)
$$

with probability one, with standard deviation,

$$
\sigma_{\hat{C}_n} = \frac{1}{\sqrt{n}} \sqrt{\text{Var}[C_i^{(bs)}]} \approx \frac{1}{\sqrt{n}} \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (C_i^{(bs)} - \hat{C}_n)^2}, \quad (7.75)
$$

where in the last term the unbiased sample variance estimate was used.

There is more to the Monte Carlo application than these basic estimates, i.e., there are variance and bias reduction techniques to improve performance and accuracy.
7.2.5. **Nonparametric, Multivariate Kernel Regression**:\(^a\)

*Kernel smoothers* are useful for smoothly fitting data to a curve or surface, particularly when the user wants to do some continuous operations on the curve, like plotting and finding optima. Whereas, splines fit smooth curves by numerical interpolation by matching values and derivatives at data points using a low degree polynomial (cubics are often used, fitting up to second derivatives or more) interpolation. The kernel smoothers are related to the kernel density estimators, except that kernel smoothing regression gives an estimation of an expectation the response scalar variable, the \( y \), relative to the explanatory \( m \)-vector, the \( \mathbf{x} \).

\(^a\)This and other sections come from Carmona (’04) Chapter 4, but the kernel smoothing part is not recommended for students.
For **independent or explanatory vector**, the distribution is represented by a **normalized kernel**, \( K_{\vec{X}} \), with common scaled bandwidth \( b_x \), the estimated smooth function for a sample of \( n \) independent observations, \( \{ \vec{X}_i, Y_i | i = 1 : n \} \), has the form

\[
y_{\text{dist}} \approx \phi(\vec{x}; b_x) = \frac{\sum_{i=1}^{n} Y_i K_{\vec{X}}\left(\frac{\vec{x} - \vec{X}_i}{b_x}\right)}{\sum_{j=1}^{n} K_{\vec{X}}\left(\frac{\vec{x} - \vec{X}_j}{b_x}\right)}, \quad (7.76)
\]

where the kernel \( K_{\vec{X}}(\vec{\xi}) \) is some model proper (i.e., integrates to one on the domain) density like normal, uniform or triangular and is used with a **standardized argument** \( \vec{\xi} \) centered about some data point \( \vec{x}_i \) and normalized with the scale of the bandwidth \( b_x \) for better computational properties. Standardized variables reduces the effects of floating point truncation errors. The normal kernel is often used because of supporting theory. Also, due to centered arguments, usually the kernel is assumed to symmetric, i.e., \( K_{\vec{X}}( -\vec{\xi}) = K_{\vec{X}}(\vec{\xi}) \).
Actually, the smoothed function $\phi(\vec{x}; b_\vec{x})$ is basically a simulation of the conditional expectation of a dependent or response variable $y$ conditioned on the independent or explanatory variable $\vec{x}$, since

$$E[Y | \vec{X} = \vec{x}] = \int_{-\infty}^{+\infty} y f_{Y | \vec{X}}(y | \vec{X} = \vec{x}) dy$$

$$= \int_{-\infty}^{+\infty} y f_{\vec{X},Y}(\vec{x}, y) dy / f_\vec{X}(\vec{x}) ,$$

by a Bayes’ rule for densities,

$$f_{Y | \vec{X}}(y | \vec{X} = \vec{x}) = \frac{f_{\vec{X},Y}(\vec{x}, y)}{f_\vec{X}(\vec{x})} ,$$

following from the definition of conditional probability\(^a\)

For motivation, consider the **univariate kernel estimation** of $f_X(x)$,

$$f_X(x) \sim \hat{f}_X(x) = \frac{1}{nb_x} \sum_{i=1}^{n} K_X\left(\frac{x - X_i}{b_x}\right). \quad (7.79)$$

Assuming that the **joint kernel is separable**, i.e.,

$$K_{X,Y}(\xi, \eta) = K_X(\xi) \cdot K_Y(\eta), \quad (7.80)$$

and that the joint density has the estimate,

$$f_{X,Y}(x, y) \sim \hat{f}_{X,Y}(x, y) = \frac{1}{nb_x b_y} \sum_{i=1}^{n} K_{X,Y}\left(\frac{x - X_i}{b_x}, \frac{y - Y_i}{b_y}\right), \quad (7.81)$$

then

$$\hat{f}_{X,Y}(x, y) = \frac{1}{nb_x b_y} \sum_{i=1}^{n} K_X\left(\frac{x - X_i}{b_x}\right) K_Y\left(\frac{y - Y_i}{b_y}\right). \quad (7.82)$$
Also, by the conditional expectations,

\[ f_X(x)E[Y|X=x] = \int_{-\infty}^{+\infty} y f_{X,Y}(x,y) dy \]

\[ \approx \frac{1}{nb_x b_y} \sum_{i=1}^{n} K_X \left( \frac{x-X_i}{b_x} \right) \]

\[ \cdot \int_{-\infty}^{+\infty} y K_Y \left( \frac{y-Y_i}{b_y} \right) dy \]

\[ = \frac{1}{nb_x} \sum_{i=1}^{n} Y_i K_X \left( \frac{x-X_i}{b_x} \right), \quad (7.83) \]

since, in the \( y \)-integral, letting \( \eta = (y-Y_i)/b_y \),

\[ \int_{-\infty}^{+\infty} y K_Y \left( \frac{y-Y_i}{b_y} \right) dy = b_y \left( Y_i + \int_{-\infty}^{+\infty} \eta K_Y(\eta) d\eta \right) = b_y Y_i, \quad (7.84) \]

by the fact that \( K_Y \) is also a symmetric and proper density like \( K_X \).
Finally by reassembling our formulas, we have the desired motivational result,

\[ E[Y|X = x] \simeq \sum_{i=1}^{n} Y_i K_X \left( \frac{x - X_i}{b_x} \right) / \sum_{j=1}^{n} K_X \left( \frac{x - X_j}{b_x} \right), \quad (7.85) \]

the denominator sum coming from using another approximation \( f_X(x) \simeq \hat{f}_X(x) \) consistent with the joint density estimated approximation.

The kernel smoothing regression formula \((7.85)\) has been implemented using a univariate Gaussian kernel by Yi Cao as \texttt{ksr.m} and posted on the \texttt{MATLAB} Central File Exchange at

\[ \text{Univariate Kernel Regression MATLAB code} \]

Cao refers to kernel estimators such as \((7.85)\) as \textbf{Nadaraya-Watson} kernel regressions.
The **multivariate kernel smoothing case** is more complicated, or rather tedious, due to dimensional complexity, even if the kernel is separable in vector variable \( \vec{x} = [x_j]_{1 \times m} \) with corresponding observations \( X = [X_{i,j}]_{n \times m} = [\vec{X}_i]_{n \times 1} \), i.e., in the separable case,

\[
K_{\vec{X}}((\vec{x} - \vec{X}_i)./\vec{b}_x) = \prod_{k=1}^{m} K_{X_k}\left(\frac{x_k - X_{i,k}}{b_{x_k}}\right).
\]  

(7.86)

for \( i = 1:n \) vector observations, where the vector bandwidth scaling is \( \vec{b}_x = [b_{x_j}]_{1 \times m} \), and ./ denotes the element by element division of MATLAB. Substituting (7.86) into (7.85) in place of the univariate kernel gives the needed objective.

\[
E[Y | \vec{X} = \vec{x}] \approx \sum_{i=1}^{n} Y_i K_{\vec{X}}((\vec{x} - \vec{X}_i)./\vec{b}_x) / \sum_{j=1}^{n} K_{\vec{X}}((\vec{x} - \vec{X}_j)./\vec{b}_x),
\]  

(7.87)

Cao also has implemented a multivariate kernel regression code, again the kernel is Gaussian kernel, and is called **ksrmv.m** at

*Multivariate Kernel Regression MATLAB code*
7.2.6. **Kernel Regression for Implied Volatility (IV) Computations:**

In the **Black-Scholes implied volatility (BSIV) inverse problem**, here starting at $t = 0$, given options data $\{K_i, T_i, S_0\}$, the match is

$$C_0^{(bs)}(M_i, T_i; \sigma_i^{(iv)}) = C_0^{(mkt)}(M_i, T_i), \quad (7.88)$$

defining $\sigma_i^{(iv)}$ for each $i = 1:n$, where the moneyness variable for $t = 0$

$$M_i \equiv S_0/K_i^a$$

for $k = 1:n$ helps to reduce the problem dimensionality. Similarly, the $C_0^{(bs)}$ and $C_0^{(mkt)}$ are defined with $t = 0$ to suppress the current time, so $\tau_i = T_i$ from (7.59). We also assume here the risk-free rate $r_i$ is taken from the current **U.S. Federal Reserve Target Rate**, so is fixed in this computation.

---

*a*Recall, sometimes the reciprocal $K/S_0$ is used for moneyness, a more suitable form for put options.
One can solve try to solve the inverse problem (7.88) directly by a root-finding method like \texttt{fzero.m} or several of the variations of Newton’s method, obtaining the estimate \( \hat{\sigma}_i^{(bsiv)}(M_i, T_i) \).

Then the kernal smoothing using the \( \hat{\sigma}_i^{(bsiv)}(M_i, T_i) \) data gives us the smoothed BSIV estimate

\[
\hat{\sigma}_{ks}^{(bsiv)}(M, T) = \sum_{i=1}^{n} \hat{\sigma}_i^{(bsiv)}(M_i, T_i) K_M\left(\frac{M - M_i}{b_M}\right) K_T\left(\frac{T - T_i}{b_T}\right) \tag{7.89}
\]

\[
/ \sum_{j=1}^{n} K_M\left(\frac{M - M_j}{b_M}\right) K_T\left(\frac{T - T_j}{b_T}\right) ,
\]

using the usual kernel smoothing procedure (7.87), designated by the \( ks \) in \( \hat{\sigma}_{ks}^{(bsiv)}(M, T) \). This estimate should be suitable for plotting an estimated BSIV volatility surface with a 2D-grid for \((M, T)\), and this can also be used for the CJD model replacing the BS model. Gaussian kernels are acceptable for both \( M \) and \( T \), so the kernel can be of the form \( K_X(x) = \text{normpdf}(x, 0; 1) \) in \texttt{MATLAB} notation.
A second approach tries to handle data error robustly by combining **Nadaraya-Watson smoothing with a least squares, approach**. A recent least squares kernel smoothing estimator is by Fengler et al. (2003)\(^a\) seemed to be more appropriate for (7.88) IV inverse problem and efficient using a robust weighting \(W(M_i)\) depending on the moneyness and regression on a weighted least squares with respect to the strike price \(K_i\) and time to maturity \(T_i\). That is, the BSIV least squares estimate.

\[
\hat{\sigma}_{ls}^{(bsiv)}(M, T) = \arg\min_{\sigma} \sum_{i=1}^{n} \left( C_i^{(mkt)}(M_i, T_i) - C_i^{(bs)}(\sigma) \right)^2 W(M_i) \\
\cdot K_M \left( \frac{M - M_i}{b_M} \right) K_T \left( \frac{T - T_i}{b_T} \right),
\]

(7.90)

where \(C_i^{(mkt)} = C^{(mkt)}(K_i, T_i)\), \(C_i^{(bs)}(\sigma) = C_0^{(bs)}(M_i, T_i; \sigma)\), \(M_i = S_0/K_i\) and \(M = S_0/K\). This can also be used for a volatility surface with 2D-grid for either BS model or CJD model.

The semi-bandwidth terms, $b_M$ and $b_T$, should be reasonable estimates of the variability of $M$ and $T$, respectively. A convenient selection of the bandwidth scaling is the bandwidth of the data, i.e.,

$$b_X = \text{std}(\vec{X}, 0; 1)$$  \hspace{1cm} (7.91)

where $\vec{X} = [X_i]_{n \times 1}$, such that $X_i$ is either $\{M_i$ or $\tilde{M}_i = 1/M_i\}$, or $T_i$. An alternate optimal formulation is that of Bowman and Azzalini (1997)\textsuperscript{a} using median values, so that

$$b_X = \text{median}(\text{abs}(\vec{X} - \text{median}(\vec{X}))) / 0.6745 \times (4/3/n)^{0.2}. \hspace{1cm} (7.92)$$

\textsuperscript{a}A.W. Bowman, A. Azzalini (1997), Applied Smoothing Techniques for Data Analysis: the Kernel Approach With S-Plus Illustrations, Oxford University Press, Oxford, UK. See also Y. Cao (2008), ksr.m, ksrlin.m or any other of Cao’s series on kernel smoothing regression in MATLAB Central for a code applications of Bowman & Azzalini’s bandwidth scaling or Nadaraya-Watson’s least squares smoothing.
A convenient weighting function, suggested by use of \texttt{robustfit.m} is the \textit{'fair'} weighting function mentioned on (6.5; L6p3), i.e.,
\begin{equation}
W(M) = \frac{1}{1+\text{abs}(M)}.
\end{equation}
(7.93)
The moneyness weights should give less weight to the less traded ITM options and the Fengler et al. (2003) suggest using
\begin{equation}
W(\tilde{M}) = \frac{\text{atan}(\pm \beta (1-\tilde{M}))}{\pi} + 0.5,
\end{equation}
(7.94)where they use the reciprocal moneyness $\tilde{M} = 1./M$ here and in the kernel, with speed control $\beta \approx 9$ and where the $(\pm)$ is the usual sign for calls $(+)$ and puts $(-)$.

One of my Singapore FINM 331 Winter 2009 students, Rudy Sitter, put his volatility surface code for Black-Scholes implied volatility on MATLAB Central as \texttt{VolSsurface.m} from one of his class projects. \textbf{BSIV Volatility Surface Code LINK}^a

^a In fact, much of the updates to this lecture comes from the feedback of the Winter 2009 students in Singapore, in particular, Stephen Huang, and other campuses. There seem to be a healthy intersection between this course and Professor R. Lee’s numerical methods class.
7.2.7. **CBOE Market Quotes:**

The market European call (or put) option price can be obtained from the **CBOE Delayed Market Quotes**, Quote Table Download page\textsuperscript{a} using an appropriate market symbol\textsuperscript{b}. The first two items listed in the first column of the quote table will be the 2-digit \textit{year} and the \textit{exercise month} followed by the strike price.

\textsuperscript{a} [http://www.cboe.com/delayedquote/QuoteTableDownload.aspx](http://www.cboe.com/delayedquote/QuoteTableDownload.aspx) (See description on how change comma-delimited format to Excel, if wanted.

\textsuperscript{b} E.G., for S&P 500 Index option SPX (Caution: some companies also use that symbol and you have to avoid getting the index itself, rather than the option.) the product specification is at [http://www.cboe.com/products/indexopts/spx_spec.aspx](http://www.cboe.com/products/indexopts/spx_spec.aspx).
The expiration date\(^a\), in case of the SPX option is the Saturday after the third Friday of the month (e.g., if 09 Feb\(^b\) is the expiration month then 21 Feb. is the expiration date.). Prices are listed in CBOE points and come with a current multiplier of $100, so for example 200.00 points is $20,000.00. SPX is a European style option as is XEO is an option on the S&P100 Index, while OEX options on that index are American, early exercise, style. Index options are different from stock options in many ways. Mileage, i.e., specifications will vary for other options.


\(^b\) A fragment of the top left corner of the SPX quote table looks like

<table>
<thead>
<tr>
<th>SPX (S&amp;P 500 INDEX)</th>
<th>826.84</th>
<th>-8.35</th>
</tr>
</thead>
<tbody>
<tr>
<td>Feb 15 2009 @ 13:45 ET LastSale</td>
<td>635.10</td>
<td>0.0</td>
</tr>
<tr>
<td>Calls</td>
<td>Net</td>
<td>Bid</td>
</tr>
<tr>
<td>09 Feb 200.00 (SPV BD-E)</td>
<td>619.50</td>
<td>622.40</td>
</tr>
</tbody>
</table>

so a market call estimate would \(C^{(mkt)}(20000, 4/252)\), counting 4 trading days due to the market Monday Holiday. Note for the SPX, there are no weekly options (ticker: JX[A,B,D,E]) and weeklys are different from monthlys and long-term versions.
<table>
<thead>
<tr>
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Figure 7.1: **CBOE Quote Table** for S&P 500 Index Options with only call option columns (put columns supressed) from page 1 of **Delayed Quote Download** page from February 15, 2009.
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Figure 7.2: **CBOE Quote Table** for S&P 500 Index Options with only call option columns from page 4 of **Delayed Quote Download** page from February 15, 2009.
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Figure 7.3: **CBOE Quote Table** for S&P 500 Index Options with only call option columns from *page 10* of **Delayed Quote Download** page from February 15, 2009.
<table>
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<tr>
<th>Date</th>
<th>Strike Price</th>
<th>Open Int</th>
<th>High</th>
<th>Low</th>
<th>Close</th>
<th>Vol</th>
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<td>0</td>
</tr>
</tbody>
</table>

Figure 7.4: **CBOE Quote Table** for S&P 500 Index Options with only call option columns from page 23 of **Delayed Quote Download** page from February 15, 2009 (*Long term LEAP options, up to 3 years*).
7.2.8. **Implied Volatility Algorithm with Kernel Regression and Numerical Inversion:**

Returning to the implied volatility computations, here is our **pseudo-algorithm**:

1. **Select an option** to study and download the quote table from the CBOE, or other exchange that allow public domain downloads, from the delayed quote page.

2. **Select a few exercise times** $T_i$, some in weeks and others in months (do not forget to convert to years, since the FRB risk-free rates are in years and that dominates the units. Also, **short exercise times are more likely to produce implied volatility smiles** (like a minimum curve), while **long times produce smirks** (like a maximum curve). If you have a **volatility surface** in mind and you should, then you will need more than a few exercise times.
3. Next select a number of strike values $K_i$, enough data to produce a respectable implied volatility curve of the $\sigma_i$ (do not forget to account for quote table scaling) versus moneyness $M_i = S_0/K_i$, where $S_0$ is the current price of the underlying asset which should be on the top of your quote table (CBOE rules).

4. Compute the model call (or put) option price data using a grid of volatility values, collected in a single index for simplicity, $\sigma_i$ for $i = 1:n$ that produce a realistic range of option prices $C_i \equiv C(K_i, T_i; \sigma_i)$ with your set of contract parameters $(K_i, T_i)$ or equivalently $(M_i, T_i)$ given $S_0$ using the Black-Scholes model (7.53) or (7.51). Else, using the Black-Scholes model as a test case, the compound-jump-diffusion (CJD) model, replacing the match difference $(C_i^{(mkt)} - C_i^{(bs)}(\sigma))$ by $(C_i^{(mkt)} - C_i^{(cjd)}(\sigma))$, where $C_i^{(cjd)}(\sigma) = \hat{C}_n$ in (7.73), using a full Monte Carlo simulation\(^a\), or the simpler partial Monte Carlo with simulated jump-part only.

\(^a\)Some estimates of jump-parameters, such as an maximum likelihood estimates (MLE) on the CJD zero-one jump daily log-return model would be needed.
5. Now form the function forming the estimated market option price curve for each value of the moneyness $M_i$ with fixed $T_i$ and other parameters using the kernel smoothing technique, using either **regular kernel smoothing regression estimation** as with 
\[ \hat{\sigma}_{ks}^{(*iv)}(M, T) \] (7.89) with $* = \text{bs}$ or cjd, or the more robust **least squares kernel smoothing regression estimation** with 
\[ \hat{\sigma}_{ls}^{(*iv)}(M, T) \] (7.90). Gaussian kernel and one of several bandwidth formulas can be used, where $X_i = M_i$ or $T_i$ and $Y_i = \hat{\sigma}_i^{(*iv)}$ or $(C_i^{(mkt)} - C_i^{(bs)}(\sigma))^2$ or $(C_i^{(mkt)} - C_i^{(cjd)}(\sigma))^2$, with other parameters suppressed. However, in particular, the user needs to keep track of the contract set $(M_i, T_i)$ each $i$-kernel smoothing operation.

---

\[ a \] See the MATLAB public domain kernel smoothing regression (KSR) code \texttt{ksr}, potentially a vector-argument kernel so could use $\bar{x}_i = [M_i, T_i]'$ with output response $Y_i$, described later. Also, other regression methods such as maximum likelihood could be used or other smoothing methods such as spline interpolation.
6. Then, for the implied volatility step is basically a nonlinear zero-finding problem: find an $\sigma^*$ such that $g(\sigma^*, \cdot) = C_i^{(\text{mkt})}$, for each fixed $i$, where $C_i^{(\text{mkt})} \equiv C^{(\text{mkt})}(K_i, T_i)$ or $C^{(\text{mkt})}(M_i, T_i)$, which in principle could determine a volatility surface. There are many basic methods that could be used here, such as the classic univariate zero finder $\text{fzero}$ for scalar function of a scalar variable that is in MATLAB or Newton’s methods or any of its quasi-variants.

7. When the roots for each $i$ data pair for $i = 1:n$ are assembled, then implied volatility curves versus moneyness and parameterized by exercise time can be plotted. Also, the implied or local volatility surface should be plotted against both moneyness and exercise or maturity time is three-dimensional graphs with surf or mesh using a 2D-grid in $(M, T)$. 
7.2.9. **Kernel Smoothing Regression (KSR)**: 

1. **Syntax:**
   
   ```
   function r = ksr(x, y, b, n),
   ```
   
   computes the Gaussian kernel regression of \( y \) versus \( x \) and outputs the structure \( r \). Part of a KSR-series with `ksrlin`, `ksrmv` ....

2. **Input:** The \( x \) is the explanatory data \( n \)-vector, \( y \) is the response data \( n \)-vector, \( b \) or \( h \) is a specified bandwidth of the kernel (if the user wants `ksr` to compute an optimal bandwidth then use \( r = ksr(x, y) \) form, and \( n \) is specified data length but should not be needed.

3. **Output:** The \( r \) is a structure, such that \( r.h \) is the computed bandwidth \( b \) or \( h \) and \( r.n \) is the **number of samples** and \( r.f(r.x) = y(x) + e \) is the form of the regression computed, all when the short form is used. The regression is plotted for the forms \( r = ksr(x, y) \) and \( r = ksr(x, y, b) \).

See the MATLAB Central Exchange for more documentation and code.

---

aKernel Smoothing Regression by Yi Cao, 2008, 

7.2.10. **Univariate Root or Zero Finder (fzero):**

1. **Syntax:**
   
   \[
   [x, fval, exitflag, output] = \texttt{fzero}(@f, x0, options);
   \]

   solves the zero or root problem for a scalar valued function \( f \) of a single scalar argument \( x \), for an \( x^* \) such that \( f(x^*) = 0 \) given a start \( x0 \) and objective function \( f \) appearing as the first argument as the pointer or handle \( @f \) usually pointing to a subfunction within the main function m-file.

2. Additional parameter can be passed to the (sub-)function \( f \) using a global statement in called and calling functions, as with \texttt{fminsearch} of Lecture 5, in fact, the syntax is much like that of \texttt{fminsearch}, except of the multivariate properties.

3. The output arguments have essentially the same descriptions as those in \texttt{fminsearch} and all but \( x \) are optional.

See D. Higham (2004) Chapters 14 and 20 for other methods for implied volatility, including Monte Carlo. *For instance, \( h(x) = g(x) - C_{i}^{(mkt)} \) and \( x0 = \sigma_i \).*
Figure 7.5: Black-Scholes implied volatility by SPX European call options of 5 different maturities, using a maximum likelihood to minimize the mean square error (MSE) between market observations and BS predictions. Option prices were quoted on April 10, 2006 (G. Yan, PhD Thesis, 2006).
Black–Scholes Implied Volatility with $T = 11$ days

- Market
- SV
- SVJD–Uniform
- SVJD–Normal
- SVJD–DbExp
Black–Scholes Implied Volatility with $T = 168$ days

- Market
- SV
- SVJD–Uniform
- SVJD–Normal
- SVJD–DbExp

Moneyness $(S/K)$

Black–Scholes Implied Volatility

FINM 331/Stat 339 W10 Financial Data Analysis — Lecture7-page66 — Floyd B. Hanson
* Reminder: Lecture 7 Homework/Project Posted in Chalk Assignments, due in PDF by Lecture 9 in Chalk Assignments!

* Summary of Lecture 7:

1. Method of Moments
2. Vanilla, European Options
3. Market Calibration and Implied Volatility
4. Risk-Neutral Pricing and Implied Volatility
5. Compound-Jump-Diffusions and Option Pricing
6. Nonparametric, Multivariate Kernel Regression
7. Kernel Regression for Implied Volatility (IV) Computations
8. CBOE Market Quotes
9. Implied Volatility (pseudo) Algorithm
10. Kernel Smoothing Regression in MATLAB
11. Hybrid Root Finder $fzero$