Lecture 1 Homework: Stochastic Jump and Diffusion Processes

(Due by Lecture 2 in Chalk FINM345 Digital Dropbox)

You must show your work, code and/or worksheet for full credit.
There are 10 points per question if correct answer.

1. Gaussian Process from Zero Mean – Unit Variance Wiener Process: Let \{t_i : t_{i+1} = t_i + \Delta t_i, i = 0 : n; t_0 = 0; t_{n+1} = T\} be a variably spaced partition of the time interval \([0, T]\) with \(\Delta t_i > 0\). Show the following properties and justify them by giving a reason for every step, such as a property of the process or a property of expectations:

   (a) Let \(G(t) = \mu t + \sigma W(t)\) and \(\Delta G(t_i) \equiv G(t_i + \Delta t_i) - G(t_i)\) with \(\mu\) and \(\sigma > 0\) constants, then show that

   \[
   E[\Delta G(t_i)] = \mu \Delta t_i, \\
   \text{Var}[\Delta G(t_i)] = \sigma^2 \Delta t_i, \\
   \text{Cov}[\Delta G(t_i), \Delta G(t_j)] = \sigma^2 \Delta t_i \delta_{i,j}
   \]

   for \(i, j = 0 : n\), where \(\delta_{i,j}\) is the Kronecker delta.

   (b) Simulate the trajectories of the processes \(G(t)\) with a professional graph, using 4 samples, with constant parameter values \(\mu = 2.448 \times 10^{-4}\), \(\sigma = 1.121 \times 10^{-2}\) and \(\Delta t = 1/252\), and \(T = 2\) is the total time interval, i.e., simulate the integral of \(dG(t) = \mu dt + \sigma dW(t)\).

2. Simple Poisson Process with Non-Unit Amplitude: Let \(\{t_i : t_{i+1} = t_i + \Delta t_i, i = 0 : n; t_0 = 0; t_{n+1} = T\}\) be a variably spaced partition of the time interval \([0, T]\) with \(\Delta t_i > 0\). Show the following properties and justify them by giving a reason for every step, such as a property of the process or a property of expectations:

   (a) Let \(H(t) = \nu P(t)\) and \(\Delta H(t_i) \equiv H(t_i + \Delta t_i) - H(t_i)\) with \(\lambda > 0\) and \(\nu > -1\) constants, then show that

   \[
   E[\Delta H(t_i)] = \nu \lambda \Delta t_i, \\
   \text{Var}[\Delta H(t_i)] = \nu^2 \lambda \Delta t_i, \\
   \text{Cov}[\Delta H(t_i), \Delta H(t_j)] = \nu^2 \lambda \Delta t_i \delta_{i,j}
   \]

   for \(i, j = 0 : n\).

   (b) Simulate the trajectories of the process \(H(t)\) with a professional graph, using 4 samples, with constant parameter values \(\lambda = 5.241\), \(\nu = 7.45 \times 10^{-3}\), \(\Delta t = 1/252\), and \(T = 2\) years is the total time interval, i.e., simulate the integral of \(dH(t) = \nu dP(t)\).

3. Wiener and Poisson Tables Partial Justification:

   (a) Derive the \(m = 3 : 4\) entries in Table 1 on page L1-p29 for \(E[(\Delta W(t))^m]\).

   (b) Derive the \(m = 3 : 4\) entries in Table 1 on page L1-p58 for \(E[(\Delta P(t))^m]\) and \(E[(\Delta P(t) - \lambda \Delta t)^m]\).
4. Integral of Wiener Differential Squared:
   
   (a) Simulate the integrals of $dX(t) = (dW(t))^2$ using four (4) samples of sample size $n = 1000$ each and time interval $T = 2$ units, plotting $X(t)$ versus $t$ on a professional graph.
   
   (b) Do the same for sample size $n = 10000$ to simulate convergence.
   
   (c) Verify the nondifferentiability of $W(t)$ by showing that $E[\Delta W(t)/\Delta t] = 0$, but $\text{Var}[\Delta W(t)/\Delta t] = 1/\Delta t \rightarrow +\infty$ as $\Delta t \rightarrow 0^+$.

5. Integral of Product of Time and Wiener Differentials:
   
   (a) Simulate the integrals of $dY(t) = dt \cdot dW(t)$ using four (4) samples of sample size $n = 1000$ each and time interval $T = 2$ units, plotting $Y(t)$ versus $t$ on a professional graph.
   
   (b) Do the same for sample size $n = 10000$ to simulate convergence.