FINM345/STAT390 Stochastic Calculus – Hanson – Autumn 2009

Lecture 3 Homework: Stochastic Jump and Diffusion Processes

(Due by Lecture 4 in Chalk FINM345 Digital Dropbox)

You must show your work, code and/or worksheet for full credit.
There are 10 points per question if correct answer.

1. **Finance Oriented Martingales.** A **martingale** in continuous time satisfies the essential property that
$$\mathrm{E}[M(t)|M(s)] = M(s),$$
for all $0 \le s < t$ provided its absolute value has finite expectation, i.e., $\mathrm{E}[|M(t)|] < \infty$ for all $t \ge 0$, plus some other technical properties.

   (a) Show that
$$M_1(t) = \ln(X(t)) - \mathrm{E}[\ln(X(t))]$$
is a martingale where $Y(t) = \ln(X(t))$ symbolically satisfies the solution to the general linear diffusion SDE transformed to state-independent SDE form (E.g., **(3.3) on L3-54 or (4.25)** in text).

2. **Exponential-Martingale Counterexample to Simple Notion that Martingales are always Driftless, a common stochastic finance legend.**

   (a) Derive the function $\beta(t)$ that makes
$$M_2 = \beta(t)X(t)$$
a martingale if $X(t)$ symbolically satisfies the linear diffusion SDE, (E.g., **(3.2) on L3-53 or (4.24)** in text).

   (b) Show that $M_2(t)$ is not driftless, i.e.,
$$\mathrm{E}[M_2(t)] \neq 0,$$
in absence of trivial initial conditions, i.e., $x_0 \neq 0$.

   {*Comment: This is a counterexample showing that if $M(t)$ is a martingale, then it in not necessarily a driftless process.* }

3. **Trigonometric Itô Integrals.** Derive the Itô stochastic integral formulas for
$$\int_0^t \cos(aW(s))dW(s) \quad \& \quad \int_0^t \sin(aW(s))dW(s)\ , \tag{1}$$
where a is a real constant $\neq 0$.

4. Solve the following (Itô) diffusion SDE for $X(t)$, $\mathrm{E}[X(t)]$, and $\mathrm{Var}[X(t)]$:
$$dX(t) = \left(aX^2(t) + b^2X^3(t)\right)dt + bX^2(t)dW(t),$$
where a and b are real constants, and $X(0) = x_0 > 0$, with probability one.
(*Hint: Seek a transformation $Y(t) = f(X(t))$ for some f such that $Y(t)$ satisfies a constant coefficient SDE.*)

5. Square Root Noise Problem. Solve the following Itô diffusion SDE for $X(t)$, $\mathrm{E}[X(t)]$, and $\mathrm{Var}[X(t)]$,

$$dX(t) = \left(a\sqrt{X(t)} + b^2/4\right) dt + b\sqrt{X(t)} dW(t) \,, \tag{2}$$

where a and b are real constants, and $X(0) = x_0 > 0$,with probability one.
{*Hint: seek a transformation* $Y(t) = F(X(t))$ *for some* F *such that* $Y(t)$ *satisfies a constant coefficient SDE. Square root noise models are often used for stochastic volatility or variance models in Finance. Warning: Homework hints are optional.*}