1. Finance Oriented Martingales. A martingale in continuous time satisfies the essential property that
\[ E[M(t)|M(s)] = M(s), \]
for all \( 0 \leq s < t \) provided its absolute value has finite expectation, i.e., \( E[|M(t)|] < \infty \) for all \( t \geq 0 \), plus some other technical properties.

(a) Show that
\[ M_1(t) = \ln(X(t)) - E[\ln(X(t))] \]
is a martingale where \( Y(t) = \ln(X(t)) \) symbolically satisfies the solution to the general linear diffusion SDE transformed to state-independent SDE form (E.g., (3.3) on L3-54 or (4.25) in text).

2. Exponential-Martingale Counterexample to Simple Notion that Martingales are always Driftless, a common stochastic finance legend.

(a) Derive the function \( \beta(t) \) that makes
\[ M_2(t) = \beta(t)X(t) \]
a martingale if \( X(t) \) symbolically satisfies the linear diffusion SDE, (E.g., (3.2) on L3-53 or (4.24) in text).

(b) Show that \( M_2(t) \) is not driftless, i.e.,
\[ E[M_2(t)] \neq 0, \]
in absence of trivial initial conditions, i.e., \( x_0 \neq 0 \).

\{ Comment: This is a counterexample showing that if \( M(t) \) is a martingale, then it in not necessarily a driftless process. \}

3. Trigonometric Itô Integrals. Derive the Itô stochastic integral formulas for
\[ \int_0^t \cos(aW(s))dW(s) \quad & \quad \int_0^t \sin(aW(s))dW(s), \] (1)
where \( a \) is a real constant \( \neq 0 \).

4. Solve the following (Itô) diffusion SDE for \( X(t), E[X(t)], \) and \( \text{Var}[X(t)] \):
\[ dX(t) = (aX^2(t) + b^2X^3(t)) \, dt + bX^2(t)dW(t), \]
where \( a \) and \( b \) are real constants, and \( X(0) = x_0 > 0 \), with probability one.

(Hint: Seek a transformation \( Y(t) = f(X(t)) \) for some \( f \) such that \( Y(t) \) satisfies a constant coefficient SDE.)
5. **Square Root Noise Problem.** Solve the following Itô diffusion SDE for $X(t)$, $E[X(t)]$, and $\text{Var}[X(t)]$,

$$dX(t) = \left(a\sqrt{X(t)} + b^2/4\right) dt + b\sqrt{X(t)}dW(t),$$

where $a$ and $b$ are real constants, and $X(0) = x_0 > 0$, with probability one.

*Hint: seek a transformation $Y(t) = F(X(t))$ for some $F$ such that $Y(t)$ satisfies a constant coefficient SDE. Square root noise models are often used for stochastic volatility or variance models in Finance. Warning: Homework hints are optional.*