You must show your work, code and/or worksheet for full credit.
There are 10 points per question if best correct answer and negative points for
missing homework sets.
Corrections are in \textcolor{red}{Red} as are comments, November 22, 2009

1. Show that the nonsingular, explicit, exact solution \[L9-p16:(9.13)]
\[V(t) = e^{-\kappa_v(t)} \left( \sqrt{V_0 + 0.5 \int_0^t e^{-\kappa_v(s)}/2 (\sigma_v dW_v(s))} \right)^2, \quad (1)\]
when \(\sigma_v^2(t) = 4 \kappa_v(t) \theta_v(t) \forall t\), is a solution satisfying the mean-reverting, square-root diffusion \[L9-p16:(9.2)]
\[dV(t) = \kappa_v(t)(\theta_v(t) - V(t)) dt + \sigma_v(t) \sqrt{V(t)} dW_v(t), \quad V(0) = V_0 > 0, \quad (2)\]
by the Itô calculus or by increment expansion of \(V(t)\) solution (9.13) in the limit of \(dt\)-precision.

2. Simulate a solution to the SV-SDE in Eq. (1) using the more robust finite difference form that is less sensitive to numerical problems for small values of the variance, i.e., use
\[V^{(\varepsilon)}(t + \Delta t) = \max \left( V^{(\varepsilon)}(t) + \kappa_v \cdot (\theta_v - V^{(\varepsilon)}(t)) \cdot \Delta t + \sigma_v \cdot \sqrt{V^{(\varepsilon)}(t)} \cdot \Delta W_v(t), \varepsilon_v \right), \quad (3)\]
\[V^{(\varepsilon)}(0) = V_0, \quad \varepsilon_v \text{ is some small positive cutoff to avoid very small variance values and } \Delta t \text{ is the simulation time step.}\]
In addition, consider the deterministic solution for constant coefficients,
\[V^{(\text{det})}(t) = V_0 \cdot \exp(-\kappa_v \cdot t) + \theta_v \cdot (1 - \exp(-\kappa_v \cdot t)), \quad V^{(\text{det})}(0) = V_0 > 0, \quad (4)\]
Let \(V_0 = 0.25 \text{ per year, } \kappa_v = 2.0 \text{ per year, } \theta_v = 0.15 \text{ per year, } \sigma_v = 0.20 \text{ per year, } T = 2 \text{ years and } N = 8000 \cdot T \text{ giving } \Delta T \text{ and } \varepsilon_v = \sqrt{\Delta t} \text{ to ensure } \Delta t / \varepsilon_v \ll 1 \text{ for } \Delta t \ll 1. \text{ Then}\]
(a) Simulate the cutoff modified solution \(V^{(\varepsilon)}(t)\) in (3) with the given parameters.
(b) Simulate the deterministic solution \(V^{(\text{det})}(t)\) in (4), i.e., with \(\sigma_v = 0\), on \([0,T]\) using the same time steps.
(c) Plot the simulations of \(V^{(\varepsilon)}(t)\) and \((\Delta SV)\), and the difference
\[\delta V(t) = V^{(\varepsilon)}(t) - V^{(\text{det})}(t)\] on a single plot with appropriate legend. Also, print out the maximum of absolute value of \(\delta V(t)\).
(d) Discuss the results. Also, justify why \(\{V_0, \kappa_v, \theta_v, \sigma_v\}\) all are in units \textit{per year} when \(t\) by convention has those units; and tell what are the units of \(\sigma_s\); the Black-Scholes stock price volatility are.