Lecture 10 (from Chicago)
Stochastic-Volatility, Jump-Diffusions: American Option Pricing and Optimal Portfolios

6:30-9:30 pm, 30 November 2009 at Kent 120 in Chicago

7:30-10:30 pm, 30 November 2009 at UBS Stamford

7:30-10:30 am, 01 December 2009 at Spring in Singapore
10. Stochastic-Volatility, Jump-Diffusion (SVJD): American Option Pricing and Optimal Portfolios:

* 10.1. SVJD American Option Pricing

* 10.1.0. SVJD American Option Pricing Outline:

1. Stochastic-Volatility Jump-Diffusion Model.
4. American Option *Linear Complementarity Problem* Finite Differences.
5. Computation and Comparison of Methods.
6. Checking Approximation with Market Data.
7. Conclusions.
10.1.1. Stochastic-Volatility Jump-Diffusion (SVJD) SDE

[Hanson and Yan (ACC2007), invited talk in honor of I. Karatzas, *Stochastic Theory and Control in Finance*]: Assume asset price \( S^{(rn)}(t) \), under a risk-neutral probability, follows a jump-diffusion process and conditional variance \( V(t) \) follows the Heston (1993) square-root mean-reverting diffusion:

\[
dS^{(rn)}(t) = S^{(rn)}(t)((r_0 - \lambda_0 \bar{\nu})dt + \sqrt{V(t)}dW_s(t)) + dCP_s(t, S^{(rn)}(t)\nu(Q)),
\]

where the compound Poisson jump process is

\[
CP_s(t, S(t)\nu(Q)) = \sum_{j=1}^{P(t)} S(T_j^-)\nu(Q_j)
\]

and

\[
dV(t) = \kappa_v(\theta_v - V(t)) dt + \sigma_v \sqrt{V(t)}dW_v(t),
\]

where \( V(t) \geq \varepsilon_v > 0 \). Here, \( r_0 \) = risk-free interest rate; \( W_s(t) \) and \( W_v(t) \) satisfy \( \text{Corr}[dW_s(t), dW_v(t)] = \rho_v(t)dt \); \( P(t) \) has intensity \( \lambda_0 \); \( \nu(Q) = \text{Poisson jump-amplitude} \); \( Q = \ln(\nu(Q) + 1) \) is the amplitude mark process.
10.1.2. Log-Uniform Jump-Diffusion Model

[Hanson and Westman (ACC2002)]:

\[ \phi_Q(q) = \frac{1}{b-a} \begin{cases} 
1, & a \leq q \leq b \\
0, & \text{else} 
\end{cases}, \quad a < 0 < b, \]

where \( \mu_j \equiv E_Q[Q] = 0.5(b+a) \) is the mark mean;

\( \sigma^2_j \equiv Var_Q[Q] = (b-a)^2/12 \) is the mark variance; and the jump-amplitude mean is

\[ \bar{\nu} \equiv E[\nu(Q)] \equiv E[e^Q - 1] = (e^b - e^a)/(b-a) - 1. \]

Finite jump-amplitudes and fat tail realism \( \implies \)

- **NYSE circuit breakers** limit extreme jumps since 1988-9;
- **In optimal portfolio problem, finite-support distributions allow realistic borrowing and short-selling** [Hanson and Zhu (Sethi2006)].
- **Uniformly distributed extreme tails.**
10.1.3 American Put Option Pricing:

{Note: American CALL option on non-dividend stock, it is not optimal to exercise before maturity; so American call price is equal to corresponding European call price, at least in the case of diffusions.}

- **American Put Option Price:**

\[
P^{(A)}(s, v, t; K, T) = \sup_{\hat{\tau}} \mathbb{E}^{(rn)} \left[ e^{-r_0 (\hat{\tau} - t)} \max[K - S(\hat{\tau}), 0] \mid S(t) = s, V(t) = v \right]
\]

on the domain \( \mathcal{D}_{s,t} = \{(s, t) \mid [0, \infty) \times [0, T]\} \), where \( K \) is the strike price, \( T \) is the maturity date, \( \mathcal{T}(t, T) \) are a set of random stopping times \( \hat{\tau} \in \mathcal{T}(t, T) \) (on the Snell envelope, Karatzas (1988) and K & Shreve (1998)) satisfying \( t < \hat{\tau} \leq T \).

- **Early Exercise Feature:** The American option can be exercised at any time \( \hat{\tau} \in [0, T] \), unlike the European option.
Hence, there exists a Critical Curve \( s = S^*(t) \), a free boundary, in the \((s, t)\)-plane, separating the domain \( \mathcal{D}_{s,t} \) into two regions:

- **Continuation Region** \( \mathcal{C} \), where it is optimal to hold the option, i.e., if \( s > S^*(t) \), then 
  \[ P^{(A)}(s, v, t; K, T) > \max[K - s, 0]. \] Here, \( P^{(A)} \) will have the same description as the European price \( P^{(E)} \).

- **Exercise Region** \( \mathcal{E} \), where it is optimal to exercise the option, i.e., if \( s \leq S^*(t) \), then 
  \[ P^{(A)}(s, v, t; K, T) = \max[K - s, 0]. \]
The American put option price satisfies a partial integro-differential equation (PIDE) similar to that of the European option price, recalling that $S(t) = s$ and $V(t) = v$, so let $P_t^{(A)}(s, v; K, T) = P_t^{(A)}(s, v, t)$, then

$$0 = P_t^{(A)}(s, v, t) + \mathcal{A} \left[ P^{(A)} \right](s, v, t)$$

$$\equiv P_t^{(A)} + (r_0 - \lambda_0 \bar{V}) s P_s^{(A)} + \kappa_v (\theta_v - v) P_v^{(A)} - r_0 P^{(A)}$$
$$+ 0.5 \left( v s^2 P_{ss}^{(A)} + 2 \rho_v \sigma_v v s P_{sv}^{(A)} + \sigma_v^2 v P_{vv}^{(A)} \right)$$
$$+ \lambda_0 \int_{-\infty}^{\infty} \left( P^{(A)}(se^q, v, t) - P^{(A)}(s, v, t) \right) \phi_Q(q) dq,$$

for $(s, t) \in \mathcal{C}$ and defining the **backward operator** $\mathcal{A}$. 
• **American put option** pricing problem as **free boundary** problem:

\[
0 = P_t^{(A)}(s, v, t) + \mathcal{A} \left[ P^{(A)} \right](s, v, t) \quad (10.4)
\]

for \((s, t) \in \mathcal{C} \equiv [S^*(t), \infty) \times [0, T] ;

\[
0 > P_t^{(A)}(s, v, t) + \mathcal{A} \left[ P^{(A)} \right](s, v, t) \quad (10.5)
\]

for \((s, t) \in \mathcal{E} \equiv [0, S^*(t)] \times [0, T] .\) where **critical stock price** \(S^*(t)\) is not known **a priori** as a function of time, **called the free boundary**.
• **Conditions in the Continuation Region** $C$:
  
  ◦ European put terminal condition limit:
    \[
    \lim_{t \to T} P^{(A)}(s, v, t; K, T) = \max[K - s, 0],
    \]
  
  ◦ Zero stock price limit of option:
    \[
    \lim_{s \to 0} P^{(A)}(s, v, t; K, T) = K,
    \]
  
  ◦ Infinite stock price limit of option:
    \[
    \lim_{s \to \infty} P^{(A)}(s, v, t; K, T) = 0,
    \]

  ◦ **Critical option value limit:**
    \[
    \lim_{s \to S^*(t)} P^{(A)}(s, v, t; K, T) = K - S^*(t),
    \]

  ◦ **Critical tangency/smooth contact limit in addition:**
    \[
    \lim_{s \to S^*(t)} \left( \frac{\partial P^{(A)}}{\partial s} \right) (s, v, t; K, T) = -1.
    \]
10.1.4 Heuristic Quadratic Approximation for American Put Options:

- **Heuristic Quadratic Approximation [MacMillan (1986)] Key Insight:** if the PIDE applies to American options $P^{(A)}$ as well as European options $P^{(E)}$ in the continuation region, it also applies to the American option optimal exercise premium,

\[ \varepsilon^{(P)}(s, v, t; K, T) \equiv P^{(A)}(s, v, t; K, T) - P^{(E)}(s, v, t; K, T), \]

where $P^{(E)}$ is given by Fourier inverse in [Yan and Hanson (2006), also Lecture 9].

- **Change in Time:** Assuming

\[ \varepsilon^{(P)}(s, v, t; K, T) \simeq G(t)Y(s, v, G(t)) \]

and choosing $G(t) = 1 - e^{-r_0(T-t)}$ as a new time variable such that $\varepsilon^{(P)} = 0$ when $G = 0$ at $t = T$. 
• After dropping the term \( r g(1 - g) Y_g \) with \( G(t) = g \) since the quadratic \( g(1 - g) \leq 0.25 \) on \([0,1]\), making \( G(t) \) a parameter instead of variable, then the \textit{quadratic approximation} of the PIDE for \( Y(s, v, g) \) is

\[
0 = + (r_0 - \lambda_0 \nu) s Y_s - \frac{r_0}{G(t)} Y + \kappa_v (\theta_v - v) Y_v + 0.5 v s^2 Y_{ss} + \rho_v \sigma_v v s Y_{sv} + 0.5 \sigma_v^2 v Y_{vv} + \lambda_0 \int_{-\infty}^{\infty} (Y(se^q, v, G(t)) - Y(s, v, G(t))) \phi_Q(q) dq,
\]

(10.6)

with \textit{quadratic approximation boundary conditions:}

\[
\lim_{s \to \infty} Y(s, v, G(t)) = 0,
\]

\[
\lim_{s \to S^*} Y(s, v, G(t)) = \left( K - S^* - P^{(E)}(S^*, v, t) \right) / G,
\]

(10.7)

\[
\lim_{s \to S^*} Y_S(s, v, G(t)) = \left( -1 - P_s^{(E)}(S^*, v, t) \right) / G.
\]
• By \textit{constant-volatility jump-diffusion (CVJD) ad hoc} approach [Bates (1996)] reformulated, we assume that the dependence on the volatility variable $v$ is weak and replace $v$ by the \textit{constant time averaged quasi-deterministic approximation of} $V(t)$:

$$
\overline{V} \equiv \frac{1}{T} \int_0^T V(t) dt = \theta_v + (V(0) - \theta_v) (1 - e^{-\kappa_v T})/ (\kappa_v T), (10.8)
$$

assuming constant $\{\kappa_v, \theta_v\}$. The PIDE (10.6) for $Y(s, v, g)$ becomes the \textbf{linear constant coefficient OIDE} for $Y(s, v, g) \to \hat{Y}(s)$, with argument suppressed parameters $G$ and $\overline{V}$,

$$
0 = + (r_0 - \lambda_0 \overline{v}) s \hat{Y}'(s) - \frac{r}{G} \hat{Y}(s) + 0.5 \overline{V} s^2 \hat{Y}''(s) + \lambda_0 \int_{-\infty}^{\infty} \left( \hat{Y}(se^q) - \hat{Y}(s) \right) \phi_Q(q) dq. \quad (10.9)
$$
• **Solution to the linear OIDE (10.9) has the power form:**

\[ \hat{Y}(s) = c_1 s^{A_1} + c_2 s^{A_2}, \]  
(10.10)

where \( c_1 = 0 \) because positive root \( A_1 \) is excluded by the vanishing boundary condition in (10.7) on \( Y \) for large \( s \).

• Substituting power form (10.10) and the uniform distribution into (10.9) for \( \hat{Y}(s) \),

\[
0 = \overline{V} A_2^2/2 + \left( r_0 - \lambda_0 \overline{v} - \overline{V}/2 \right) A_2 - r_0/G \\
+ \lambda \left( (e^{bA_2} - e^{aA_2}) / ((b - a) A_2) - 1 \right),
\]  
(10.11)

which is a **nearly-quadratic nonlinear equation for values of interest.**
The last two boundary conditions in (10.7) give the equations satisfied by $S^*(t)$ and $c_2$. Then $S^* = S^*(t)$ can be calculated by fixed point iteration method with the expression:

$$S^* = \frac{A_2 \left( K - P^{(E)} \left( S^*, \overline{V}, t; K, T \right) \right)}{A_2 - 1 - P_S^{(E)} \left( S^*, \overline{V}, t; K, T \right)} \quad (10.12)$$

and

$$c_2 = \frac{K - S^* - P^{(E)} \left( S^*, \overline{V}, t; K, T \right)}{G \cdot \left( S^* \right)^{A_2}}.$$
10.1.5 Linear Complementarity Problem (LCP) Finite Differences for American Put Options:

- Free boundary problem is transferred to partial integro-differential complementarity problem (PIDCP) formulated as follows

\[
\begin{align*}
P^{(A)}(s, v, t; K, T) - F(s) &\geq 0, \\
P^{(A)}_\tau - AP^{(A)} &\geq 0, \\
\left(P^{(A)}_\tau - AP^{(A)}\right)(P^{(A)} - F) &= 0,
\end{align*}
\] (10.13)

where \( F(s) \equiv \max[K - s, 0] \) is the put payoff function and \( \tau \equiv T - t \) is the time-to-go.
• **Crank-Nicolson second-order numerical scheme** with discrete state backward operator $L \simeq A$,

$$P^{(A)}(S_i, V_j, T - \tau_k; K, T) \equiv U(S_i, V_j, \tau_k) \simeq U_{i,j}^{(k)},$$

$$U^{(k)} = \begin{bmatrix} U_{i,j}^{(k)} \end{bmatrix},$$

$$P_T^{(A)} \simeq \frac{U^{(k+1)} - U^{(k)}}{\Delta \tau},$$

$$A P^{(A)} \simeq 0.5 L \left( U^{(k+1)} + U^{(k)} \right).$$

• **Standard Linear Algebraic Definitions:** Let

$$\hat{U}^{(k)} = \begin{bmatrix} \hat{U}_{i}^{(k)} \end{bmatrix},$$

the single subscripted version of 2D-array $U^{(k)} = \begin{bmatrix} U_{i,j}^{(k)} \end{bmatrix}$, with corresponding variables $\hat{F}$, $\hat{L}$, $\hat{M}$ and $\hat{b}^{(k)}$, so $\hat{M} \equiv I - \frac{\Delta \tau}{2} \hat{L}$ and $\hat{b}^{(k)} \equiv \left( I + \frac{\Delta \tau}{2} \hat{L} \right) \hat{U}^{(k)}$. 
• **Discretized LCP** [Cottle et al. (1992); Wilmott et al. (1995, 1998)]:

\[
\hat{U}^{(k+1)} - \hat{F} \geq 0, \quad \hat{M} \hat{U}^{(k+1)} - \hat{b}^{(k)} \geq 0,
\]

\[
\left( \hat{U}^{(k+1)} - \hat{F} \right) ^\top \left( \hat{M} \hat{U}^{(k+1)} - \hat{b}^{(k)} \right) = 0,
\]  

(10.14)

• **Projective Successive OverRelaxation (PSOR)** (PSOR ≡ Projected SOR algorithm, projected onto the max function) with SOR acceleration parameter \( \omega \) for LCP (10.14) by iterating \( \tilde{U}^{(n+1)}_i \) for \( \hat{U}^{(k+1)}_i \) until changes are sufficiently small:

\[
\tilde{U}^{(n+1)}_i = \max \left( \hat{F}_i, \tilde{U}^{(n)}_i + \omega \hat{M}^{-1} \left( \hat{b}^{(k)}_i - \sum_{j<i} \hat{M}_{i,j} \tilde{U}^{(n+1)}_j \right) - \sum_{j\geq i} \hat{M}_{i,j} \tilde{U}^{(n)}_j \right),
\]

where the sum splitting over iterates is from SOR.
• **Full Boundary Conditions for** \( U(s, v, \tau) \):

\[
U(0, v, \tau) = F(0) \quad \text{for} \quad v \geq 0 \quad \text{and} \quad \tau \in [0, T],
\]

\[
U(s, v, \tau) \rightarrow 0 \quad \text{as} \quad s \rightarrow \infty \quad \text{for} \quad v \geq 0 \quad \text{and} \quad \tau \in [0, T],
\]

\[
U(s, 0, \tau) = F(s) \quad \text{for} \quad s \geq 0 \quad \text{and} \quad \tau \in [0, T],
\]

\[
U_v(s, v, \tau) = 0 \quad \text{as} \quad v \rightarrow \infty \quad \text{for} \quad s \geq 0 \quad \text{and} \quad \tau \in [0, T].
\]

• **Initial Condition for** \( U(s, v, \tau) \):

\[
U(s, v, 0) = F(s) \quad \text{for} \quad s \geq 0 \quad \text{and} \quad v \geq 0.
\]
• **Discretization of the PIDCP:** The first-order and second-order spatial derivatives and the cross-derivative term are all approximated with the *standard second-order accurate finite differences*, using a nine-point computational molecule.

• **Linear interpolation** is applied to the *jump integral* term and *quadratic extrapolation* of the solution is used for the *critical stock price* $S^*(t)$ calculation, with comparable accuracy.
10.1.6 Computation and Comparison of Methods for American Put Options:

- The **Heuristic Quadratic Approximation** and **LCP/PSOR** approaches for American put option pricing are **implemented and compared**. All computations are done on a 2.40GHz Celeron\(^R\) CPU. For the quadratic approximation analytic formula, one American put option price and critical stock price can be computed in about 7 seconds. The finite difference method can give a series of option prices for different stock prices and maturity for a specific strike price by one implementation. A single implementation, with \(51 \times 101 \times 51\) grids and acceleration parameter \(\omega = 1.35\), takes 17 seconds.
• The American put option prices for Parameters: 
  \( r_0 = 0.05, S_0 = \$100 \); the stochastic volatility part: 
  \( V = 0.01, \kappa_v = 10, \theta_v = 0.012, \sigma_v = 0.1, \rho_v = -0.7; \)
  and the uniform jump part: \( a = -0.10, b = 0.20 \) and \( \lambda_0 = 0.5. \)
American & European Put Option Price for T = 0.1

Option Prices, P(A) & P(E)
Moneyness, S/K

American, P(A)
European, P(E)

(a) American and European put option prices for \( T = 0.1 \) years.

(b) American and European put option prices for \( T = 0.25 \) years.

Figure 10.1: The **heuristic quadratic approximation** gives SVJD-Uniform American \( P^{(A)} = P_{QA}^{(A)} \) compared to European \( P^{(E)} \) put option prices for \( T = 0.1 \) (\( \simeq 5 \) weeks) and \( 0.25 \) years (3 months), with averaged approximation of \( V(t) \).
(a) American and European put option prices for $T = 0.5$ years.

(b) Critical stock prices for $T = 0.5$.

Figure 10.2: The **heuristic quadratic approximation** gives SVJD-Uniform American $P^{(A)} = P_{QA}^{(A)}$ compared European $P^{(E)}$ put option prices and **critical stock prices** for $T = 0.5$ years, with averaged approximation of $V(t)$. 
(a) American put option prices by LCP. 

(b) Critical stock prices for K = 100.

Figure 10.3: **PSOR finite difference implementation of LCP** gives SVJD-Uniform American put option prices $U(S, V, \tau) = P_{LCP}^{(A)}$ and critical stock prices $S^*(\tau; V)$ (using quadratic extrapolation approximations for smooth contact to the payoff function).
Figure 10.4: Comparison of American put option prices evaluated by quadratic approximation (QA) and LCP finite difference (LCPFD) methods when \( S = $100 \) and \( V = 0.01 \) (\( \sqrt{V} = 0.1 \)). Maximum price difference \( P_{QA}^{(A)} - P_{LCP}^{(A)} = \{0.08, 0.14, 0.21\} \) for \( T = \{0.1, 0.25, 0.5\} \) years, respectively, so QA is probably good for practical purposes.
10.1.7 Checking Quadratic Approximation with Market Data:

- Choose same time XEO (European options) and OEX (American options) quotes on April 10, 2006 from CBOE. They are based on same underlying S&P 100 Index.
- Use XEO put option quotes to estimate parameter values of the European put option pricing for the quadratic approximation.
- Calculate American put option prices by quadratic approximation formula with estimated parameter values and compare the results with OEX quotes. Mean square error, \( \text{MSE} = 0.137 \), is obtained, showing good fit.
Table 1: **SVJD-Uniform Parameters Estimated** from XEO quotes on April 10, 2006

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa_v$</td>
<td>10.62</td>
</tr>
<tr>
<td>$\theta_v$</td>
<td>0.0136</td>
</tr>
<tr>
<td>$\sigma_v$</td>
<td>0.175</td>
</tr>
<tr>
<td>$\rho$</td>
<td>-0.547</td>
</tr>
<tr>
<td>$a$</td>
<td>-0.140</td>
</tr>
<tr>
<td>$b$</td>
<td>0.011</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>0.549</td>
</tr>
<tr>
<td>$V$</td>
<td>0.0083</td>
</tr>
<tr>
<td>MSE</td>
<td>0.195</td>
</tr>
</tbody>
</table>
Figure 10.5: American put option price differences between quadratic approximation (QA) and OEX (put) quotes, when $S = \$100$ and $V = 0.01$ ($\sqrt{V} = 0.1$). Maximum absolute price difference: $P_{QA}^{(A)} - P_{OEX}^{(A)} = \{\$0.41, \$0.46, \$0.73, \$1.15, \$0.68\}$ for $T = \{11, 39, 67, 102, 168\}$ days, respectively.
Figure 10.6: Critical stock prices $S^*$ using QA versus K with OEX quote data, when $S = $100 and $V = 0.01$ ($\sqrt{V} = 0.1$) for $T = \{11, 39, 67, 102, 168\}$ days.
10.1.8 Conclusions for American Put Options:

- An alternative stochastic-volatility jump-diffusion (SVJD) stock model is proposed with square root mean reverting for stochastic-volatility combined with log-uniform jump amplitudes.
- The heuristic quadratic approximation (QA) and the accurate LCP finite difference scheme for American put option pricing are compared, with QA being good and fast for practical purposes.
- The QA results are also checked against real market American option pricing data OEX (with XEO for Euro. price base), yielding reasonable results considering the simplicity of QA.
10.2. SVJD Optimal Portfolio and Consumption Problem

10.2.0. SVJD Optimal Portfolio and Consumption Problem Outline:

1. Introduction.
2. Optimal Portfolio Problem and Underlying SVJD Model.
4. CRRA Canonical Solution to Optimal Portfolio Problem.
5. Computational Results.
6. Conclusions.
10.2.1. Introduction to SVJD Extension of Merton Portfolio Optimization Problem:

{Note: Some of the beginning of this part repeats somethings of the 10.1, first part of L10, but that is for completeness.}

- **Merton pioneered the optimal portfolio and consumption problem** for geometric diffusions used HARA (hyperbolic absolute risk-aversion) utility in his lifetime portfolio [Merton, RES (1969)] and general portfolio [Merton, JET (1971)] papers. However, there were some errors, in particular with bankruptcy boundary conditions and vanishing consumption.
10.2.2. Underlying Stochastic-Volatility, Jump-Diffusion (SVJD) Return Model: [Hanson (BFS2008)]

Stock Price Linear Stochastic Differential Equation (SDE):

\[
dS(t) = S(t)(\mu_s(t)dt + \sqrt{V(t)}dG_s(t)) + dCP_s(t, S(t)\nu(Q)),
\]

(10.15)

where the compound Poisson jump process is

\[
CP_s(t, S(t)\nu(Q)) = \sum_{j=1}^{P(t)} S(T_j^-)\nu_s(Q_j),
\]

and

- \( S(t) = \text{stock price}, \ S(0) = S_0 > 0; \)
- \( \mu_s(t) = \text{expected rate of return} \) in absence of asset jumps;
- \( V(t) = \text{stochastic variance} = (\text{stochastic volatility})^2 = \sigma_s^2(t); \)
• \( G_s(t) = \text{stock price diffusion process} \), normally distributed such that \( \mathbb{E}[dG_s(t)] = 0 \) and \( \text{Var}[dG_s(t)] = dt \), replacing \( dW_s(t) \) since wealth process \( W(t) \) has priority for symbol \( W \);

• \( P_s(t) = \text{Poisson jump counting process} \), Poisson distributed such that \( \mathbb{E}[dP_s(t)] = \lambda_s(t)dt = \text{Var}[dP_s(t)] \);

• \( \nu_s(v, t, q) = \text{Poisson jump-amplitude} \) with underlying random mark variable \( q = Q \), selected for log-return so that \( Q = \ln(1 + \nu_s(v, t, Q)) \), such that \( \nu_s(v, t, q) > -1 \);

• \( T_k^- \) is the pre-jump time and \( Q_k \) is an independent and identically distributed (IID) mark realization at the \( k \)th jump;

• Processes \( dG_s(t) \) and \( P_s(t) = P_s(t; Q) \) along with \( Q_k \) are independent, except that \( Q_k \) is conditioned on a jump-event at \( T_k \).
Stochastic-Volatility (Square-Root Diffusion) Model:
[CIR, Econometrica (1985); Heston, RFS (1993); FPS, book (2000)]

\[
dV(t) = \kappa_v(t) (\theta_v(t) - V(t)) \, dt + \sigma_v(t) \sqrt{V(t)} \, dG_v(t),
\]

with

- \( V(t) \geq \min(V(t)) > 0^+ \), \( V(0) = V_0 \geq \min(V(t)) > 0^+ \);
- \textit{Log-rate} \( \kappa_v(t) > 0 \); \textit{reversion-level} \( \theta_v(t) > 0 \); \textit{volatility of volatility (variance)} \( \sigma_v(t) > 0 \);
- \( G_v(t) = \text{variance diffusion process} \), normally distributed such that
  \( \mathbb{E}[dG_v(t)] = 0 \) and \( \text{Var}[dG_v(t)] = dt \), with \textit{correlation}
  \( \text{Corr}[dG_s(t), dG_v(t)] = \rho(t) \, dt \);
- Note: SDE (10.16) is singular for transformations as \( V(t) \to 0^+ \)
due to the square root, unlike SDE (10.1) for \( S(t) \) where the
singularity is removable through the log transformation, but
Itô-Taylor chain rule or simulation applications might not be valid unless
\( \Delta t \ll \sqrt{\min(V(t))} \ll 1 \).
Double-Uniform Jump-Amplitude Q Mark Distribution:

\[ \Phi_Q(q; v, t) = p_1(v, t) \frac{q - a(v, t)}{|a|(v, t)} I\{a(v, t) \leq q < 0\} + \left(p_1(v, t) + p_2(v, t) \frac{q}{b(v, t)}\right) I\{0 \leq q < b(v, t)\} + I\{b(v, t), \leq q < \infty\}, \quad q \in [a(v, t), b(v, t)], \]

where \( a(v, t) < 0 < b(v, t), \ p_1(v, t) + p_2(v, t) = 1, \)

- **Mark Mean:**
  \[ \mu_j(v, t) \equiv E_Q[Q] = (p_1(v, t)a(v, t) + p_2(v, t)b(v, t))/2; \]

- **Mark Variance:**
  \[ \sigma_j^2(v, t) \equiv \text{Var}_Q[Q] = (p_1(v, t)a^2(v, t) + p_2(v, t)b^2(v, t))/3 - \mu_j^2(v, t); \]

- **More motivation:** Double-uniform distribution unlinks the different extreme behaviors in crashes and rallies.
Wealth Portfolio with Bond, Stock and Consumption:

- **Portfolio**: Riskless asset or bond at price $B(t)$ and Risky asset or stock at price $S(t)$ (10.15), with instantaneous portfolio change fractions $U_b(t)$ and $U_s(t)$, respectively, such that $U_b(t) = 1 - U_s(t)$.

- **Exponential Bond Price Process**:
  \[ dB(t) = r(t)B(t)dt, \quad B(0) = B_0. \]

- **SVJD Portfolio Wealth Process $W(t)$, Less Consumption $C(t)$ with Self-Financing**: 
  \[ dW(t) = W(t)(r(t)dt + U_s(t)((\mu_s(t) - r(t)))dt + \sqrt{V(t)}dG_s(t)) + dC P_s(t, W(t)U_s(t)\nu(Q)) - C(t)dt, \]
  subject to constraints $W(0) = W_0 > 0$, $W(t) > 0$, $v = V(t) > 0$, $0 < C(t) \leq C_0^{(\text{max})}(v, t)W(t)$ and $U_0^{(\text{min})}(v, t) \leq U_s(t) \leq U_0^{(\text{max})}(v, t)$, while allowing extra shortselling ($U_s(t) < 0$) and extra borrowing ($U_b(t) < 0$).
10.2.3. SVJD Portfolio Optimal Objective — The Maximal, Expected Utilities of Final Wealth and Running Consumption:

\[ e^{-\beta(t)} J^*(w, v, t) = \max_{\{u,c\}} \left[ E \left[ e^{-\beta(t_f)} U_w(W(t_f)) \right] \right. \]

\[ + \left. \int_t^{t_f} e^{-\beta(\tau)} U_c(C(\tau)) d\tau \right] \]

(10.18)

where

- **Cumulative Discount back to** \( t = 0 \): \( \beta(t) = \int_0^t \beta(\tau) d\tau \), where \( \beta(t) \) is the instantaneous discount rate. The \( t_f = T \) is the final time.

- **Consumption and Final Wealth Utility Functions**: \( U_c(c) \) and \( U_w(w) \) are bounded, strictly increasing and concave.

- **Variable Classes**: State variables are \( w \) and \( v \), while control variables are \( u \) and \( c \).

- **Final Condition**: \( J^*(w, v, t_f) = U_w(w) \).
Absorbing Natural Boundary Condition:

Approaching bankruptcy as \( w \to 0^+ \), then, by the consumption constraint, as \( c \to 0^+ \) and by the objective \((10.18)\),

\[
e^{-\beta(t)} J^*(0^+, v, t) = \mathcal{U}_w(0^+) e^{-\beta(t_f)} + \mathcal{U}_c(0^+) \int_t^{t_f} e^{-\beta(s)} ds. \tag{10.19}
\]

- This is the simple variant what Merton gave as a correction in his 1990 book for his 1971 optimal portfolio paper.
- However, [Karatzas, Lehoczky, Sethi and Shreve (KLASS) (1986) and [Sethi and Taksar (1988)] pointed out that it was necessary to enforce the non-negativity of wealth and consumption.
Derivation of Stochastic Dynamic Programming PIDE by Stochastic Calculus:

Assume that the optimization and expectation of state and control stochastic processes can be decomposed into independent increments over nonoverlapping time intervals by Bellman’s Principle of Optimality [Hanson (2007), Ch. 6 & Ex. 6.3], so that

\[ e^{-\beta(t)} J^*(w, v, t) = \max_{\{U, C\}(t,t+\Delta t)} \left[ E\{G, CP_Q\}(t,t+\Delta t) \right. \]
\[ \int_t^{t+\Delta t} e^{-\beta(\tau)} U_c(C(\tau)) d\tau \]
\[ + e^{-\beta + \Delta \beta}(t) \]
\[ \cdot J^*(((W + \Delta W)(t), (V + \Delta V)(t), t+\Delta t)) \]
\[ \left| W(t) = w, V(t) = v, U(t) = u, C(t) = c \right]. \]

Next, the limit is taken using the stochastic calculus.
As $\Delta t \to 0^+$, we simplify the state $S\Delta E$ notation as

$$\Delta W \triangleq \mu_w \Delta t + \sigma_w \Delta G_s + \nu_w \Delta P_s, \nu_w = uw(\exp(Q) - 1)$$
and

$$\Delta V \triangleq \mu_v \Delta t + \sigma_v \Delta G_v, \text{ while using } J^* = J^*(w, v, t) \text{ and conditional values, so}$$

$$e^{-\bar{\beta}(t)} J^*(w, v, t) \triangleq \max_{\{u, c\}} \left[ e^{-\bar{\beta}(t)} \left( U_c(c) \Delta t + J^* + \Delta t \left( -\beta(t) J^* + J_t^* + J^*_w \mu_w + J^*_v \mu_v \\ + 0.5 J^*_{ww} \sigma^2_w + \rho_w \sigma_w \hat{\sigma}_v J^*_{wv} + 0.5 J^*_v \hat{\sigma}_v^2 \\
+ \lambda_s \int_Q dq \phi_Q(q) \cdot (J^*(w + u(e^q - 1)w, v, t) - J^*(w, v, t)) \right) \right] \right].$$

Cancellation of $e^{-\bar{\beta}(t)} J^*(w, v, t)$ on both sides and $\Delta t$, yields

$$0 = \max_{\{u, c\}} \left[ U_c(c) - \beta(t) J^* + J_t^* + J^*_w \mu_w + J^*_v \mu_v \\
+ 0.5 J^*_{ww} \sigma^2_w + \rho_w \sigma_w \hat{\sigma}_v J^*_{wv} + 0.5 J^*_v \hat{\sigma}_v^2 \\
+ \lambda_s (t) \int_Q dq \phi_Q(q) (J^*(w + u(e^q - 1)w, v, t) - J^*(w, v, t)) \right].$$

(10.21)

Next, we substitute for temporary coefficients and take the maximum (*)
* 10.2.4. SVJD Portfolio Stochastic Dynamic Programming PIDE for Double-Uniform Qs:

\[
0 = J_t^*(w, v, t) - \beta(t)J^*(w, v, t) + U_c(c^*) - c^* J^*_w(w, v, t) \\
+ (r(t) + (\mu_s(t) - r(t))u^*) w J^*_w(w, v, t) \\
+ \kappa_v(t)(\theta_v(t) - v)J^*_v(w, v, t) + \frac{1}{2}v(u^*)^2w^2 J^*_{ww}(w, v, t) \\
+ \frac{1}{2}\sigma_v^2(t)v J^*_{vv}(w, v, t) + \rho_v(t)\sigma_v(t)vu^*w J^*_{wv}(w, v, t) \\
+ \lambda_s(t)\left(\frac{p_1(v,t)}{|a(v,t)|}\int_a(v,t) + \frac{p_2(v,t)}{b(v,t)}\int_b(v,t)\right) \\
\cdot \left(J^*((1+(e^q - 1)u^*)w, v, t) - J^*(w, v, t)\right) dq,
\]

where \( u^* = u^*(w, v, t) \in [U_0^{(\text{min})}(v, t), U_0^{(\text{max})}(v, t)] \) and \( c^* = c^*(w, v, t) \in [0, C_0^{(\text{max})}(v, t)w] \) are the optimal controls, if they exist, while \( J^*_w(w, v, t) \) and \( J^*_{ww}(w, v, t) \) are the continuous partial derivatives with respect to wealth \( w \). Note that \((1+(e^q - 1)u^*(w, v, t))w\) is a post-jump wealth argument.
10.2.5. Positivity of Wealth with Jump Distribution:

Since \( (1 + (e^q - 1)u^*(w, v, t))w \) is a wealth argument in (10.23), it must satisfy the wealth positivity condition, so

\[
K(u, q) \equiv 1 + (e^q - 1)u > 0
\]
on \([a(v, t), b(v, t)]\) of the jump-amplitude density \( \phi_Q(q; v, t) \).

Lemma 10.1 **Bounds on Optimal Stock Fraction due to Positivity of Wealth Jump Argument:**

(a) If the support of \( \phi_Q(q; v, t) \) is the finite interval \( q \in [a(v, t), b(v, t)] \) with \( a(v, t) < 0 < b(v, t) \), then \( u^*(w, v, t) \) is restricted by (10.23) to

\[
-1 \frac{1}{\nu_s(v, t, b(v, t))} < u^*(w, v, t) < -1 \frac{1}{\nu_s(v, t, a(v, t))},
\]

where \( \nu_s(v, t, q) = \exp(q) - 1 \).

(b) If the support of \( \phi_Q(q; v, t) \) is fully infinite, i.e., \( (-\infty, +\infty) \), then \( u^*(w, v, t) \) is restricted by (10.23) to

\[
0 < u^*(w, v, t) < 1.
\]
Remarks: Non-Negativity of Wealth and Jump Distribution:

- Recall that \( u \) is the stock fraction, so that short-selling and borrowing will be overly restricted in the infinite support case (10.25) where \( a(v, t) \to -\infty \) and \( b(v, t) \to +\infty \), unlike the finite case (10.24), where \(-\infty < a(v, t) < 0 < b(v, t) < +\infty\).

- So, unlike option pricing, finite support of the mark density makes a big difference in the optimal portfolio and consumption problem!

- Thus, it would not be practical to use either normally or double-exponentially distributed marks in the optimal portfolio and consumption problem with a bankruptcy condition.

- If \([a_{\text{min}}, b_{\text{max}}] = [\min_t (a(v, t)), \max_t (b(v, t))]\), then the overall \( u^* \) range for the S&P500 data used is

\[
[u_{\text{min}}, u_{\text{max}}] = [-18, +12] \subset \left( \frac{-1}{(e^{b_{\text{max}}}-1)} , \frac{+1}{(1-e^{a_{\text{min}}})} \right).
\]

- Extreme tail ordering:

\[
\exp(-x^2) \ll |x|^N \exp(-|x|) \ll |x|^{-N} \ll 1, \ |x| \gg 1, \ N > 0.
\]
10.2.6. Unconstrained Optimal or Regular Control Policies:

In absence of control constraints and in presence of sufficient differentiability, the dual policy, implicit critical conditions are

- **Regular Consumption** $c^{(\text{reg})}(w, v, t) \{\text{Implicitly}\}$:
  \[
  \mathcal{U}_c'(c^{(\text{reg})}(w, v, t)) = J^*_w(w, v, t). \tag{10.26}
  \]

- **Regular Portfolio Fraction** $u^{(\text{reg})}(w, v, t) \{\text{Implicitly}\}$:
  \[
  vw^2 J^*_{ww}(w, v, t) u^{(\text{reg})}(w, v, t) = - (\mu_s(t) - r(t)) w J^*_w(w, v, t)
  - \rho \sigma_v(t) vw J^*_{wv}(w, v, t)
  - \lambda(t) w \left( \frac{p_1(v, t)}{|a|(v, t)} \int_a^0 + \frac{p_2(v, t)}{b(v, t)} \int_0^{b(v, t)} \right)
  - (e^q - 1) J^*_w(t, K(u^{(\text{reg})}(w, v, t), q) w) dq. \tag{10.27}
  \]
* 10.2.6. CRRA Utilities Canonical Solution to Optimal Portfolio Problem:

- **Constant Relative Risk-Aversion (CRRA⊂HARA) Power Utilities:**
  \[ U_c(x) = U(x) = U_w(x) = \begin{cases} 
  x^{\gamma}/\gamma, & \gamma \neq 0 \\
  \ln(x), & \gamma = 0 
\end{cases}, \quad x \geq 0, \gamma < 1. \quad (10.28) \]

- **Relative Risk-Aversion (RRA):**
  \[ RRA(x) \equiv -U''(x)/(U'(x)/x) = (1 - \gamma) > 0, \quad \gamma < 1, \]
  i.e., negative of ratio of marginal to average change in marginal utility \( U'(x) > 0 \) & \( U''(x) < 0 \) is a constant.

- **CRRA Canonical Separation of Variables:**
  \[ J^*(w, v, t) = U(w)J_0(v, t), \quad J_0(v, t_f) = 1, \quad (10.29) \]
  i.e., if valid, then wealth state dependence is known and only the time-variance dependent factor \( J_0(v, t) \) need be determined.
10.2.6. Canonical Simplifications with CRRA Utilities:

- **Regular Consumption Control is Linear in Wealth:**
  \[
  c^{(\text{reg})}(w, v, t) = w \cdot c_0^{(\text{reg})}(v, t) \equiv w/J_0^{1/(1-\gamma)}(v, t),
  \]
  where \(c_0^{(\text{reg})}(v, t)\) is a wealth fraction, with optimal consumption,
  \[
  c^*_0(v, t) = \max \left[ \min \left[ c_0^{(\text{reg})}(v, t), C_0^{(\text{max})}(v, t) \right], 0 \right]
  \]
  per \(w\).

- **Regular Fraction Control is Independent of Wealth:**
  \[
  u^{(\text{reg})}(w, v, t) = u_0^{(\text{reg})}(v, t) = \frac{1}{(1-\gamma)v} \left( \mu_s(t) - r(t) + \frac{\rho \sigma_v(t) v J_0(v, t)}{J_0(v, t)} + \lambda_s(t) I_1(u_0^{(\text{reg})}(v, t), v, t) \right),
  \]
  where \(v > 0\), in fixed point form, where
  \[
  u^* = u_0^*(v, t) = \max \left[ \min \left[ u_0^{(\text{reg})}(v, t), U_0^{(\text{max})} \right], U_0^{(\text{max})} \right],
  \]
  and
  \[
  I_1(u, v, t) \equiv \left( \frac{p_1(v, t)}{|a|(v, t)} \int_a^{b(v, t)} + \frac{p_2(v, t)}{b(v, t)} \int_0^{b(v, t)} \right) (e^q - 1) K^{\gamma-1}(u, q) dq.
  \]
10.2.7. CRRA Time-Variance Dependent Component in Formal “Bernoulli” PDE ($\gamma \neq 0; \gamma < 1$):

$$0 = J_{0,t}(v, t) + (1 - \gamma) \left( g_1 J_0 + g_2 J_0^{-1} \right)(v, t)$$

$$+ g_3(v, t) J_{0,v}(v, t) + \frac{1}{2} \sigma_v^2(t)v J_{0, vv}(v, t),$$

(10.32)

where

- **Bernoulli Coefficients** $g_1(v, t), g_2(v, t),$ and $g_3(v, t)$:

  $g_1(v, t) = g_1(v, t; u^*_0(v, t)),$

  $g_2(v, t) = g_2(v, t; c^*_0(v, t), c^{(reg)}_0(v, t)),$ and $g_3(v, t) =

  g_3(v, t; u^*_0(v, t)),$ introduce implicit nonlinear dependence

  on $u^*_0(v, t), c^*_0(v, t)$ and $c^{(reg)}_0(v, t),$ so iterations are required.

- **Formal (Implicit) Solution using Bernoulli transformation**, $J_0(v, t) = y^{1-\gamma}(v, t),$ improving iterations:

  $$0 = y_t(v, t) + g_1(v, t)y(v, t) + g_4(v, t), \quad y(v, t_f) = 1,$$

$$J_0(v, t) = \left[ e^{\bar{g}_1(v, t, t_f)} + \int_t^{t_f} g_4(v, \tau) e^{\bar{g}_1(v, t, \tau)}d\tau \right]^{1-\gamma}.$$  

(10.33)
Here,

\[
g_1(v, t) \equiv \frac{1}{1 - \gamma} \left( \beta(t) + \gamma (r(t) + (\mu_s(t) - r(t))u_0^*(v, t)) \right. \\
- \frac{1}{2} \left( 1 - \gamma \right) v(u_0^*)^2(v, t) + \lambda_s(t) \left( I_2(u_0^*(v, t), v, t) - 1 \right) \\
- \frac{1}{2} (1 - \gamma) v(u_0^*)^2(v, t) + \lambda_s(t) \left( I_2(u_0^*(v, t), v, t) - 1 \right),
\]

\[
\bar{g}_1(v, t, \tau) \equiv \int_t^\tau g_1(v, s) ds.
\]

\[
I_2(u, v, t) \equiv \left( \frac{p_1(v, t)}{|a|}(v, t) \int_0^a + \frac{p_2(v, t)}{b(v, t)} \int_0^{b(v, t)} \right) K^\gamma(u, q) dq,
\]

\[
g_2(v, t) \equiv \frac{1}{1 - \gamma} \left( \left( \frac{c_0^*(v, t)}{c_0^{(reg)}(v, t)} \right)^\gamma - \gamma \left( \frac{c_0^*(v, t)}{c_0^{(reg)}(v, t)} \right) \right),
\]

\[
g_3(v, t) = +\kappa_v(t)(\theta_v(t) - v) + \gamma \rho \sigma_v(t) v u_0^*(v, t),
\]

\[
g_4(v, t) = g_2(v, t) + g_3(v, t) y_v(v, t) + \frac{1}{2} \sigma_v^2(t) v (y_{vv} - \gamma((y_v)^2/y)) (v, t).
\]
10.2.8. CRRA Time-Variance Dependent Component in Formal “Bernoulli” PDE ($\gamma = 0$; Kelly Criterion):

**Famous Users: Ed Thorp, Warren Buffett, George Soros.**

In this medium risk-averse case of the logarithmic CRRA utility, the formal, implicit canonical solution has two terms,

$$ J^*(w, v, t) = \ln(w) J_0(v, t) + J_1(v, t), \quad (10.34) $$

with final boundary conditions $J_0(v, t) = 1$ and $J_1(v, t) = 0$.

The regular controls satisfy,

$$ c^{(\text{reg})}(w, v, t) = wc_0^{(\text{reg})}(v, t) \equiv w/J_0(v, t), $$

$$ u^{(\text{reg})}(w, v, t) = u_0^{(\text{reg})}(v, t) $$

$$ \equiv \frac{1}{v} \left( \mu_s(t) - r(t) + \rho\sigma_v(t)(J_{0, v}/J_0)(v, t) \right) $$

$$ + \lambda_s(t) I_1 \left( u_0^{(\text{reg})}(v, t), v, t \right), \quad v > 0. $$
10.2.9. Computational Considerations and Results:

- **Computational Considerations:**
  
  - The primary problem is having stable computations and much smaller time-steps $\Delta t$ are needed compared to variance-steps $\Delta V$, since the computations are *drift-dominated* over the diffusion coefficient, in that the mesh coefficient associated with $J_{0,v}$ can be hundreds times larger than that associated with $J_{0,vv}$ for the variance-diffusion.
  
  - *Drift-upwinding* is needed so the finite differences for the drift-partial derivatives follow the sign of the drift-coefficient, while central differences are sufficient for the diffusion partials.
  
  - *Iteration calculations in time, controls and volatility* are sensitive to small and negative deviations, as well as the form of the iteration in terms of the formal implicitly-defined solutions.
Results for Regular \( u^{(\text{reg})}(v_p, t) \) and Optimal \( u^*(v_p, t) \) Portfolio Fraction Policies, \( \sigma_p = \sqrt{v_p} = 16\% \):

(a) Regular fraction policy \( u^{(\text{reg})}(v_p, t) \).  
(b) Optimal fraction policy, \( u^*(v_p, t) \).

Figure 10.7: Regular and optimal portfolio stock fraction policies, \( u^{(\text{reg})}(v_p, t) \) and \( u^*(v_p, t) \) on \( t \in [1999.0, 2001.0] \), while \( u^*(v_p, t) \in [-18, 12] \).
Results for Optimal Value $J^* (w, v_p, t)$ and Optimal Consumption $c^* (w, v_p, t)$, Portfolio Fraction Policies, $\sigma_p = \sqrt{v_p} = 16\%$:

(a) Optimal portfolio value $J^* (w, v_p, t)$. (b) Optimal consumption policy $c^* (w, v_p, t)$. Figure 10.8: Optimal portfolio value $J^* (w, v_p, t)$ and optimal consumption policy $c^* (w, v_p, t)$ for $(w, v_p, t) \in [0, 110] \times [1999.0, 2001.0]$, while $c^* (w, v_p, t) \in [0, 0.75 \cdot w]$ is enforced near $t = 2001$. 
Results for Optimal Value $J^*(w_p, v, t)$ and Optimal Consumption $c^*(w_p, v, t)$, $w_p = 55$:

(a) Optimal portfolio value $J^*(w_p, v, t)$.  
(b) Optimal consumption $c^*(w_p, v, t)$.

Figure 10.9: Optimal portfolio value $J^*(w_p, v, t)$ and optimal consumption $c^*(w_p, v, t)$ at $w_p = 55$ for $(v, t) \in \times [v_{\text{min}}, 1.0] \times [1999.0, 2001.0]$, while $c^*(w_p, v, t) \in [0, 0.75 \cdot w_p]$ is enforced near $t = 2001$. 
Results for Optimal Portfolio Fraction $u^*(v, t)$:

Figure 10.10: Optimal portfolio fraction policy $u^*(v, t)$ for $(v, t) \in \times [v_{\min}, 1.0] \times [1999.0, 2001.0]$, while $u^*(v, t) \in [-18, 12]$ is enforced near small variance $v = v_{\min} > 0$. 
10.2.10. Conclusions for SVJD Optimal Portfolio and Consumption Problem:

- Generalized the optimal portfolio and consumption problem for jump-diffusions to include stochastic volatility/variance.
- Confirmed significant effects on variation of instantaneous stock fraction policies due to time-dependence of interest and discount rates for SVJD optimal portfolio and consumption models.
- Showed jump-amplitude distributions with compact support are much less restricted on short-selling and borrowing compared to the infinite support case in the SVJD optimal portfolio and consumption problem.
- Noted that the CRRA reduced canonical optimal portfolio problem is strongly drift-dominated for sample market parameter values over the diffusion terms, so at least first order drift-upwinding is essential for stable Bernoulli PDE computations.
Summary of Lecture 10?

1.

2.

3.

4.

5.