FinM 345/Stat 390 Stochastic Calculus, Autumn 2009

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Lecture 10 (from Chicago)

Stochastic-Volatility, Jump-Diffusions: American Option Pricing and Optimal Portfolios

6:30-9:30 pm, 30 November 2009 at Kent 120 in Chicago

7:30-10:30 pm, 30 November 2009 at UBS Stamford

7:30-10:30 am, 01 December 2009 at Spring in Singapore

FinM 345 Stochastic Calculus:

10. Stochastic-Volatility, Jump-Diffusion (SVJD): American Option Pricing and Optimal Portfolios:

• 10.1. SVJD American Option Pricing

∗ 10.1.0. SVJD American Option Pricing Outline:

1. Stochastic-Volatility Jump-Diffusion Model.

2. American Put Option Pricing.

3. *Heuristic Quadratic Approximation* for American Put Option.

4. American Option *Linear Complementarity Problem* Finite Differences.

5. Computation and Comparison of Methods.

6. Checking Approximation with Market Data.

7. Conclusions.

$*$ 10.1.1. Stochastic-Volatility Jump-Diffusion (SVJD) SDE

[Hanson and Yan (*ACC2007*), invited talk in honor of I. Karatzas, *Stochastic Theory and Control in Finance*]: Assume asset price $S^{(\mathrm{rn})}(t)$, under a risk-neutral probability, follows a jump-diffusion process and conditional variance $V(t)$ follows the Heston (1993) square-root mean-reverting diffusion:

$$\begin{aligned} dS^{(\mathrm{rn})}(t) = S^{(\mathrm{rn})}(t)\big((r_0 - \lambda_0\overline{\nu})dt + \sqrt{V(t)}dW_s(t)\big) \\ + d\mathrm{CP}_s(t, S^{(\mathrm{rn})}(t)\nu(Q)), \end{aligned} \tag{10.1}$$

where the compound Poisson jump process is $\mathrm{CP}_s(t, S(t)\nu(Q)) = \sum_{j=1}^{P(t)} S(T_j^-)\nu(Q_j)$ and

$$dV(t) = \kappa_v(\theta_v - V(t))\,dt + \sigma_v\sqrt{V(t)}dW_v(t), \tag{10.2}$$

where $V(t) \geq \varepsilon_v > 0$. Here, $r_0 =$risk-free interest rate; $W_s(t)$ and $W_v(t)$ satisfy $\mathrm{Corr}[dW_s(t), dW_v(t)] = \rho_v(t)dt$; $P(t)$ has intensity λ_0; $\nu(Q) =$Poisson jump-amplitude; $Q = \ln(\nu(Q)+1)$ is the amplitude mark process.

$*$ 10.1.2. Log-Uniform Jump-Diffusion Model

[Hanson and Westman (ACC2002)]:

$$\phi_Q(q) = \frac{1}{b-a}\begin{Bmatrix} 1, & a \le q \le b \\ 0, & else \end{Bmatrix}, \quad a < 0 < b,$$

where $\mu_j \equiv \mathrm{E}_Q[Q] = 0.5(b+a)$ is the mark mean; $\sigma_j^2 \equiv \mathrm{Var}_Q[Q] = (b-a)^2/12$ is the mark variance; and the jump-amplitude mean is

$$\overline{\nu} \equiv \mathrm{E}[\nu(Q)] \equiv \mathrm{E}[e^Q - 1] = (e^b - e^a)/(b-a) - 1.$$

Finite jump-amplitudes and fat tail realism $\Longrightarrow$

- **NYSE *circuit breakers*** limit extreme jumps since 1988-9;
- **In optimal portfolio problem, finite-support distributions allow realistic borrowing and short-selling** [Hanson and Zhu (Sethi2006)].
- **Uniformly distributed extreme tails.**

$*$ 10.1.3 American Put Option Pricing:

{*Note: American CALL option on non-dividend stock, it is not optimal to exercise before maturity; so American call price is equal to corresponding European call price, at least in the case of diffusions.*}

- **American Put Option Price:**

$$P^{(A)}(s,v,t;K,T) = \sup_{\widehat{\tau}} \Big[\mathrm{E}^{(\mathrm{rn})} \Big[e^{-r_0(\widehat{\tau}-t)} \max[K - S(\widehat{\tau}), 0] \\ \Big| S(t) = s, V(t) = v \Big] \Big]$$

  on the domain $\mathcal{D}_{s,t} = \{(s,t) \mid [0,\infty) \times [0,T]\}$, where K is the strike price, T is the maturity date, $\mathcal{T}(t,T)$ are a set of random stopping times $\widehat{\tau} \in \mathcal{T}(t,T)$ (*on the Snell envelope, Karatzas (1988) and K & Shreve (1998)*) satisfying $t < \widehat{\tau} \le T$.

- **Early Exercise Feature:** The American option can be exercised at any time $\widehat{\tau} \in [0,T]$, unlike the European option.

- Hence, there exists a **Critical Curve** $s = S^*(t)$, a free boundary, in the (s, t)-plane, separating the domain $\mathcal{D}_{s,t}$ into two regions:
    - **Continuation Region** $\mathcal{C}$, where it is optimal to hold the option, i.e., if $s > S^*(t)$, then $P^{(A)}(s, v, t; K, T) > \max[K - s, 0]$. Here, $P^{(A)}$ will have the same description as the European price $P^{(E)}$.
    - **Exercise Region** $\mathcal{E}$, where it is optimal to exercise the option, i.e., if $s \leq S^*(t)$, then $P^{(A)}(s, v, t; K, T) = \max[K - s, 0]$.

- The **American put option price satisfies a partial integro-differential equation (PIDE) similar to that of the European option price**, recalling that $S(t)=s$ and $V(t)=v$, so let $P_t^{(A)}(s,v,t;K,T)=P_t^{(A)}(s,v,t)$, then

$$\begin{aligned}
0 &= P_t^{(A)}(s,v,t)+\mathcal{A}\left[P^{(A)}\right](s,v,t) \\
&\equiv P_t^{(A)}+(r_0-\lambda_0\overline{\nu})sP_s^{(A)}+\kappa_v(\theta_v-v)P_v^{(A)}-r_0P^{(A)} \\
&\quad +0.5\left(vs^2P_{ss}^{(A)}+2\rho_v\sigma_v vsP_{sv}^{(A)}+\sigma_v^2vP_{vv}^{(A)}\right) \\
&\quad +\lambda_0\int_{-\infty}^{\infty}\left(P^{(A)}(se^q,v,t)-P^{(A)}(s,v,t)\right)\phi_Q(q)dq,
\end{aligned} \tag{10.3}$$

for $(s,t)\in\mathcal{C}$ and defining the **backward operator $\mathcal{A}$**.

- **American put option** pricing problem as **free boundary problem:**

$$0 = P_t^{(A)}(s,v,t) + \mathcal{A}\left[P^{(A)}\right](s,v,t) \quad (10.4)$$

for $(s,t) \in \mathcal{C} \equiv [S^*(t), \infty) \times [0,T]$;

$$0 > P_t^{(A)}(s,v,t) + \mathcal{A}\left[P^{(A)}\right](s,v,t) \quad (10.5)$$

for $(s,t) \in \mathcal{E} \equiv [0, S^*(t)] \times [0,T]$. where **critical stock price** $S^*(t)$ is not known ***a priori*** as a function of time, **called the free boundary**.

- **Conditions in the Continuation Region $\mathcal{C}$:**
  - European put terminal condition limit:
$$\lim_{t \longrightarrow T} P^{(A)}(s, v, t; K, T) = \max[K - s, 0],$$
  - Zero stock price limit of option:
$$\lim_{s \longrightarrow 0} P^{(A)}(s, v, t; K, T) = K,$$
  - Infinite stock price limit of option:
$$\lim_{s \longrightarrow \infty} P^{(A)}(s, v, t; K, T) = 0,$$
  - **Critical option value limit:**
$$\lim_{s \longrightarrow S^*(t)} P^{(A)}(s, v, t; K, T) = K - S^*(t),$$
  - **Critical tangency/smooth contact limit in addition:**
$$\lim_{s \longrightarrow S^*(t)} \frac{\partial P^{(A)}}{\partial s}(s, v, t; K, T) = -1.$$

* 10.1.4 Heuristic Quadratic Approximation for American Put Options:

- **Heuristic Quadratic Approximation [MacMillan (1986)] Key Insight**: if the PIDE applies to American options $P^{(A)}$ as well as European options $P^{(E)}$ in the continuation region, it also **applies to the American option optimal exercise premium**,
  $\varepsilon^{(P)}(s,v,t;K,T) \equiv P^{(A)}(s,v,t;K,T) - P^{(E)}(s,v,t;K,T)$,
  where $P^{(E)}$ is given by Fourier inverse in [Yan and Hanson (2006), also Lecture 9].
- **Change in Time:** Assuming
  $$\varepsilon^{(P)}(s,v,t;K,T) \simeq G(t)Y(s,v,G(t))$$
  and choosing $G(t) = 1 - e^{-r_0(T-t)}$ as a new time variable such that $\varepsilon^{(P)} = 0$ when $G = 0$ at $t = T$.

- After **dropping the term $rg(1-g)Y_g$ with $G(t)=g$ since the quadratic $g(1-g)\le 0.25$ on [0,1]**, making $G(t)$ a parameter instead of variable, then the ***quadratic approximation*** of the PIDE for $Y(s,v,g)$ is

$$\begin{aligned} 0= &+(r_0-\lambda_0\overline{\nu})\,sY_s-\frac{r_0}{G(t)}Y+\kappa_v(\theta_v-v)Y_v \\ &+0.5vs^2Y_{ss}+\rho_v\sigma_v vsY_{sv}+0.5\sigma_v^2 vY_{vv} \\ &+\lambda_0\int_{-\infty}^{\infty}\left(Y(se^q,v,G(t))-Y(s,v,G(t))\right)\phi_Q(q)dq, \end{aligned} \tag{10.6}$$

with **quadratic approximation boundary conditions:**

$$\begin{aligned} \lim_{s\to\infty} Y(s,v,G(t)) &= 0, \\ \lim_{s\to S^*} Y(s,v,G(t)) &= \left(K-S^*-P^{(E)}(S^*,v,t)\right)\Big/G, \\ \lim_{s\to S^*} Y_S(s,v,G(t)) &= \left(-1-P_S^{(E)}(S^*,v,t)\right)\Big/G. \end{aligned} \tag{10.7}$$

- By **constant-volatility jump-diffusion (CVJD)** ad hoc approach [Bates (1996)] reformulated, we assume that the dependence on the volatility variable v is weak and replace v by the **constant time averaged quasi-deterministic approximation of $V(t)$:**

$$\overline{\overline{V}} \equiv \frac{1}{T}\int_0^T \overline{V}(t)dt = \theta_v + (V(0) - \theta_v)\left(1 - e^{-\kappa_v T}\right)/(\kappa_v T), \quad (10.8)$$

assuming constant $\{\kappa_v, \theta_v\}$. The PIDE (10.6) for $Y(s, v, g)$ becomes the **linear constant coefficient OIDE** for $Y(s, v, g) \to \widehat{Y}(s)$, with argument suppressed parameters G and $\overline{\overline{V}}$,

$$0 = +(r_0 - \lambda_0 \overline{\nu})\, s\widehat{Y}'(s) - \frac{r}{G}\widehat{Y}(s) + 0.5\overline{\overline{V}}s^2\widehat{Y}''(s)$$

$$+\lambda_0 \int_{-\infty}^{\infty}\left(\widehat{Y}(se^q) - \widehat{Y}(s)\right)\phi_Q(q)dq. \quad (10.9)$$

- **Solution to the linear OIDE (10.9) has the power form**:
$$\widehat{Y}(s) = c_1 s^{A_1} + c_2 s^{A_2}, \tag{10.10}$$
where $c_1 = 0$ because positive root A_1 is excluded by the vanishing boundary condition in (10.7) on Y for large s.

- Substituting power form (10.10) and the **uniform distribution** into (10.9) for $\widehat{Y}(s)$,
$$\begin{aligned} 0 = &\overline{\overline{V}} A_2^2/2 + \left(r_0 - \lambda_0\overline{\nu} - \overline{\overline{V}}/2\right) A_2 - r_0/G \\ &+ \lambda\left(\left(e^{bA_2} - e^{aA_2}\right)/\left((b-a)A_2\right) - 1\right), \end{aligned} \tag{10.11}$$
which is a **nearly-quadratic nonlinear equation for values of interest**.

- The last two boundary conditions in (10.7) give the equations satisfied by $S^*(t)$ and c_2. Then $S^* = S^*(t)$ can be calculated by fixed point iteration method with the expression:

$$S^* = \frac{A_2\left(K - P^{(E)}\left(S^*, \overline{\overline{V}}, t; K, T\right)\right)}{A_2 - 1 - P_S^{(E)}\left(S^*, \overline{\overline{V}}, t; K, T\right)} \quad (10.12)$$

and

$$c_2 = \frac{K - S^* - P^{(E)}\left(S^*, \overline{\overline{V}}, t; K, T\right)}{G \cdot (S^*)^{A_2}}.$$

$*$ 10.1.5 Linear Complementarity Problem (LCP) Finite Differences for American Put Options:

- **Free boundary problem** is transferred to **partial integro-differential complementarity problem (PIDCP)** formulated as follows

$$\begin{aligned} P^{(A)}(s,v,t;K,T)-F(s) &\ge 0, \\ P^{(A)}_{\tau}-\mathcal{A}P^{(A)} &\ge 0, \\ \left(P^{(A)}_{\tau}-\mathcal{A}P^{(A)}\right)\left(P^{(A)}-F\right) &= 0, \end{aligned} \tag{10.13}$$

where $F(s)\equiv\max[K-s,0]$ is the **put payoff function** and $\tau\equiv T-t$ is the **time-to-go**.

- **Crank-Nicolson second-order numerical scheme** with discrete state backward operator $L \simeq \mathcal{A}$,

$$P^{(A)}(S_i, V_j, T-\tau_k; K, T) \equiv U(S_i, V_j, \tau_k) \simeq U_{i,j}^{(k)},$$

$$U^{(k)} = \left[U_{i,j}^{(k)}\right],$$

$$P_\tau^{(A)} \simeq \frac{U^{(k+1)} - U^{(k)}}{\Delta\tau},$$

$$\mathcal{A}P^{(A)} \simeq 0.5L\left(U^{(k+1)} + U^{(k)}\right).$$

- **Standard Linear Algebraic Definitions:** Let $\widehat{\mathrm{U}}^{(k)} = \left[\widehat{U}_i^{(k)}\right]$, the single subscripted version of 2D-array $U^{(k)} = \left[U_{i,j}^{(k)}\right]$, with corresponding variables $\widehat{\mathrm{F}}$, $\widehat{L}$, $\widehat{M}$ and $\widehat{\mathrm{b}}^{(k)}$, so $\widehat{M} \equiv I - \frac{\Delta\tau}{2}\widehat{L}$ and $\widehat{\mathrm{b}}^{(k)} \equiv \left(I + \frac{\Delta\tau}{2}\widehat{L}\right)\widehat{\mathrm{U}}^{(k)}$.

- **Discretized LCP [Cottle et al. (1992); Wilmott et al. (1995, 1998)]:**

$$\widehat{\mathrm{U}}^{(k+1)} - \widehat{\mathrm{F}} \geq 0, \qquad \widehat{M}\widehat{\mathrm{U}}^{(k+1)} - \widehat{\mathrm{b}}^{(k)} \geq 0,$$
$$\left(\widehat{\mathrm{U}}^{(k+1)} - \widehat{\mathrm{F}}\right)^{\top}\left(\widehat{M}\widehat{\mathrm{U}}^{(k+1)} - \widehat{\mathrm{b}}^{(k)}\right) = 0, \tag{10.14}$$

- **Projective Successive OverRelaxation (PSOR) (PSOR≡Projected SOR algorithm, projected onto the max function)** with **SOR acceleration parameter ω** for LCP (10.14) by iterating $\widetilde{U}_i^{(n+1)}$ for $\widehat{U}_i^{(k+1)}$ until changes are sufficiently small:

$$\widetilde{U}_i^{(n+1)} = \max\left(\widehat{F}_i,\ \widetilde{U}_i^{(n)} + \omega \widehat{M}_{i,i}^{-1}\left(\widehat{b}_i^{(k)} - \sum_{j<i} \widehat{M}_{i,j}\widetilde{U}_j^{(n+1)} - \sum_{j\geq i} \widehat{M}_{i,j}\widetilde{U}_j^{(n)}\right)\right),$$

where the sum splitting over iterates is from SOR.

- **Full Boundary Conditions for $U(s,v,\tau)$:**

$$U(0,v,\tau)=F(0) \text{ for } v\ge 0 \text{ and } \tau\in[0,T],$$

$$U(s,v,\tau)\to 0 \text{ as } s\to\infty \text{ for } v\ge 0 \text{ and } \tau\in[0,T],$$

$$U(s,0,\tau)=F(s) \text{ for } s\ge 0 \text{ and } \tau\in[0,T],$$

$$U_v(s,v,\tau)=0 \text{ as } v\to\infty \text{ for } s\ge 0 \text{ and } \tau\in[0,T].$$

- **Initial Condition for $U(s,v,\tau)$:**

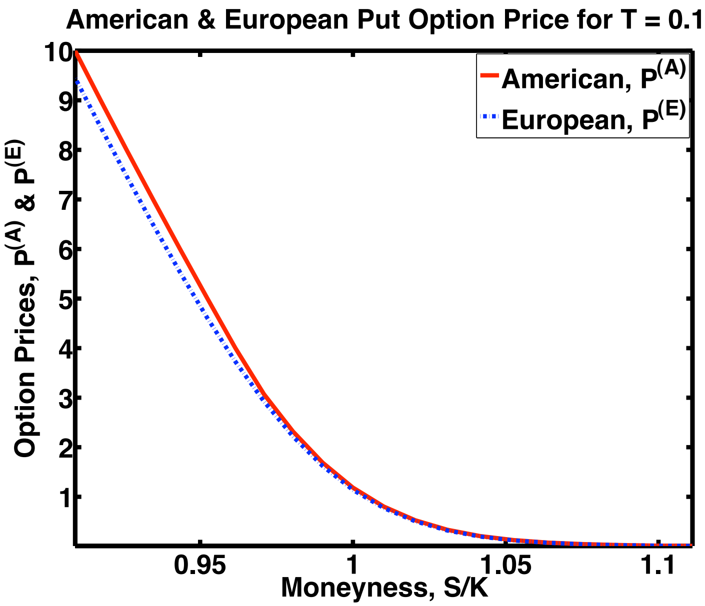
$$U(s,v,0)=F(s) \text{ for } s\ge 0 \text{ and } v\ge 0.$$

- **Discretization of the PIDCP:** The first-order and second-order spatial derivatives and the cross-derivative term are all approximated with the **standard second-order accurate finite differences, using a nine-point computational molecule**.
- **Linear interpolation** is applied to the **jump integral** term and **quadratic extrapolation** of the solution is used for the **critical stock price** $S^*(t)$ calculation, with comparable accuracy.

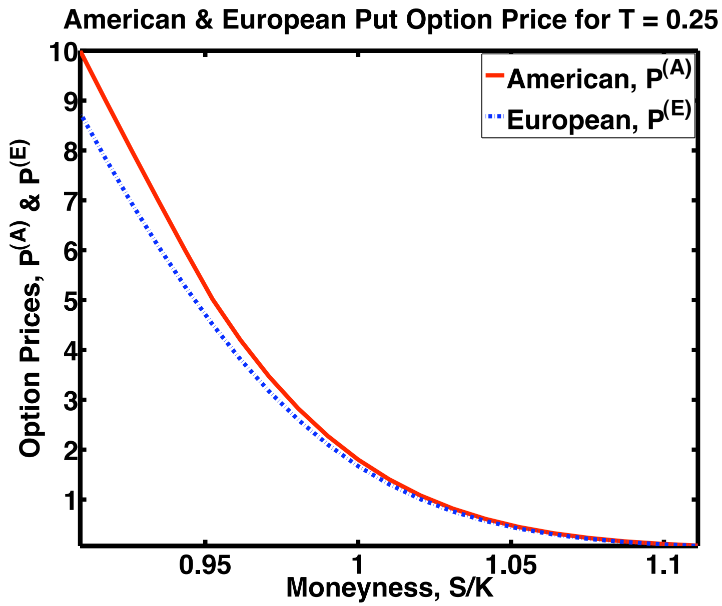
$*$ 10.1.6 Computation and Comparison of Methods for American Put Options:

- The **Heuristic Quadratic Approximation** and **LCP/PSOR** approaches for American put option pricing are **implemented and compared**. All computations are done on a 2.40GHz Celeron$^{(R)}$ CPU. For the quadratic approximation analytic formula, one American put option price and critical stock price can be computed in about 7 seconds. The finite difference method can give a series of option prices for different stock prices and maturity for a specific strike price by one implementation. A single implementation, with $51 \times 101 \times 51$ grids and acceleration parameter $\omega = 1.35$, takes 17 seconds.

- The American put option prices for **Parameters**: $r_0 = 0.05$, $S_0 = \$100$; the stochastic volatilitypart: $V = 0.01$, $\kappa_v = 10$, $\theta_v = 0.012$, $\sigma_v = 0.1$, $\rho_v = -0.7$; and the uniform jump part: $a = -0.10$, $b = 0.20$ and $\lambda_0 = 0.5$.

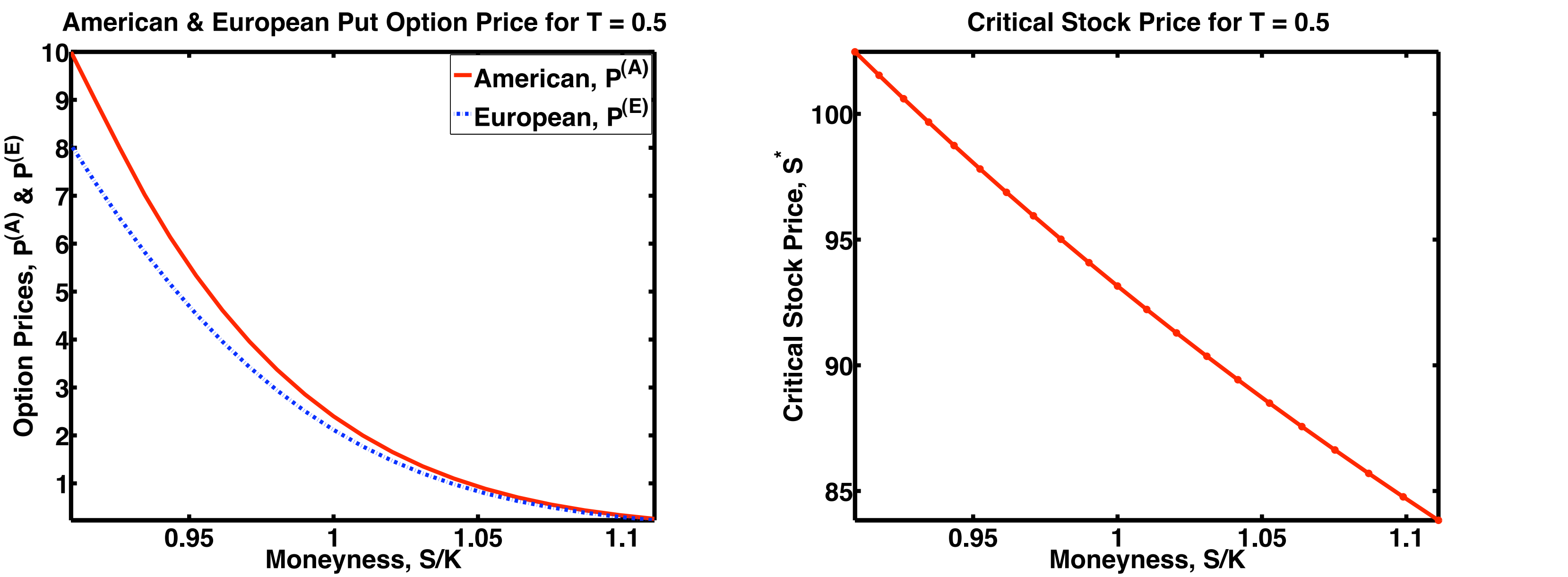


(a) American and European put option prices for $\boldsymbol{T = 0.1}$ years.



(b) American and European put option prices for $\boldsymbol{T = 0.25}$ years

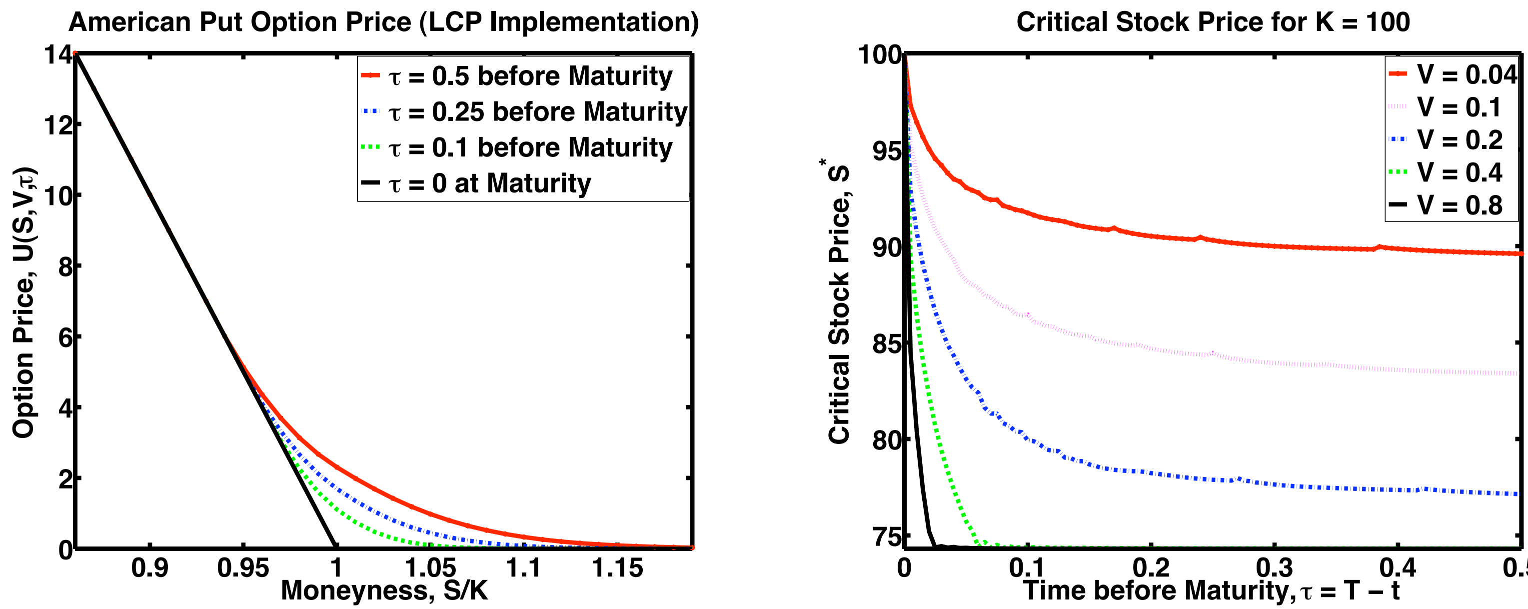
Figure 10.1: The **heuristic quadratic approximation** gives SVJD-Uniform American $\boldsymbol{P^{(A)} = P^{(A)}_{QA}}$ compared to European $\boldsymbol{P^{(E)}}$ put option prices for $\boldsymbol{T = 0.1}$ ($\simeq$ 5 weeks) and $\boldsymbol{0.25}$ years (3 months), with averaged approximation of $\boldsymbol{V(t)}$.



(a) American and European put option prices for $T = 0.5$ years.

(b) Critical stock prices for $T = 0.5$.

Figure 10.2: The **heuristic quadratic approximation** gives SVJD-Uniform American $P^{(A)} = P_{QA}^{(A)}$ compared European $P^{(E)}$ put option prices and **critical stock prices** for $T = 0.5$ years, with averaged approximation of $V(t)$.



(a) American put option prices by LCP.   (b) Critical stock prices for K = 100.

Figure 10.3: **PSOR finite difference implementation of LCP** gives SVJD-Uniform American put option prices $U(S,V,\tau){=}P^{(A)}_{LCP}$ and critical stock prices $S^*(\tau;V)$ (using quadratic extrapolation approximations for smooth contact to the payoff function).

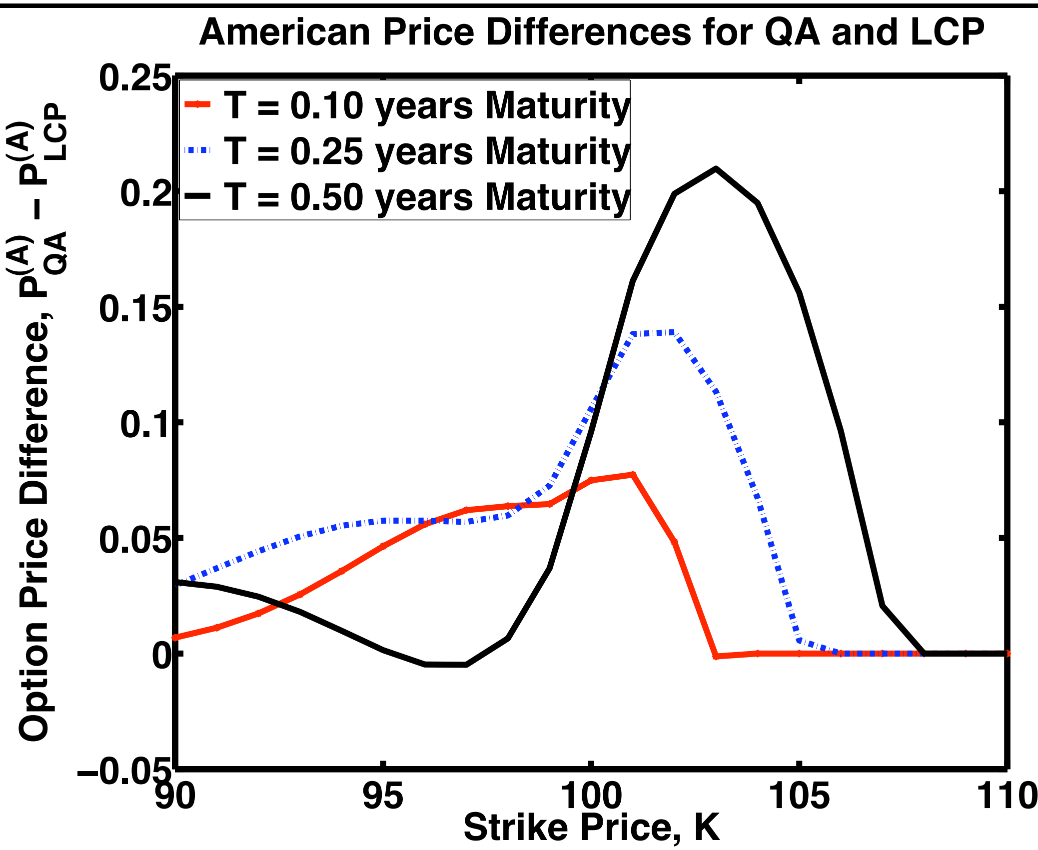


Figure 10.4: **Comparison of American put option prices evaluated by quadratic approximation (QA) and LCP finite difference (LCPFD) methods** when $S = \$100$ and $V = 0.01$ ($\sqrt{V} = 0.1$). **Maximum price difference** $P_{QA}^{(A)} - P_{LCP}^{(A)} = \{\$0.08, \$0.14, \$0.21\}$ for $T = \{0.1, 0.25, 0.5\}$ years, respectively, so QA is probably good for practical purposes.

* 10.1.7 Checking Quadratic Approximation with Market Data:

- Choose same time **XEO (European options)** and **OEX (American options)** quotes on April 10, 2006 from CBOE. They are based on same underlying S&P 100 Index.
- Use XEO put option quotes to estimate parameter values of the European put option pricing for the quadratic approximation.
- Calculate American put option prices by quadratic approximation formula with estimated parameter values and compare the results with OEX quotes. Mean square error, $\mathbf{MSE = 0.137}$, is obtained, showing good fit.

Table 1: **SVJD-Uniform Parameters Estimated** from XEO quotes on April 10, 2006

Parameter	Values
κ_v	10.62
θ_v	0.0136
σ_v	0.175
ρ	-0.547
a	-0.140
b	0.011
λ	0.549
V	0.0083
MSE	0.195

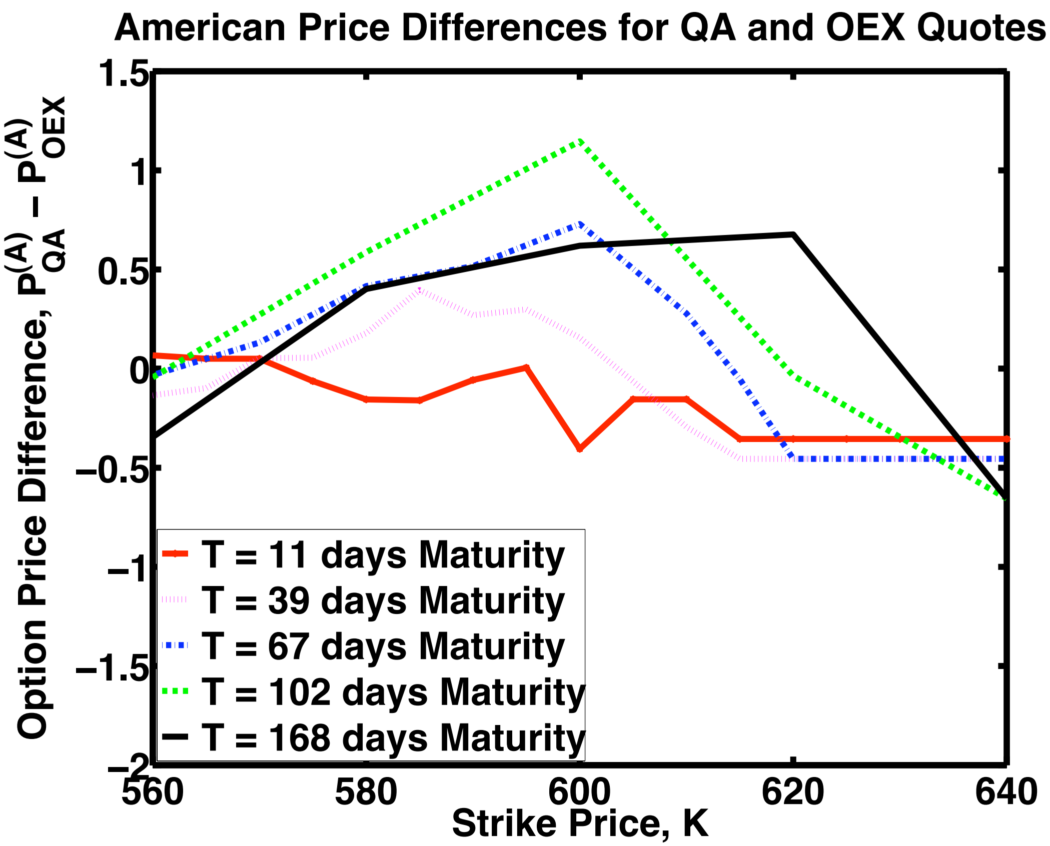


Figure 10.5: **American put option price differences between quadratic approximation (QA) and OEX (put) quotes**, when $S = \$100$ and $V = 0.01$ ($\sqrt{V} = 0.1$). **Maximum absolute price difference:** $P^{(A)}_{QA} - P^{(A)}_{OEX} = \{\$0.41, \$0.46, \$0.73, \$1.15, \$0.68\}$ for $T = \{11, 39, 67, 102, 168\}$ days, respectively.

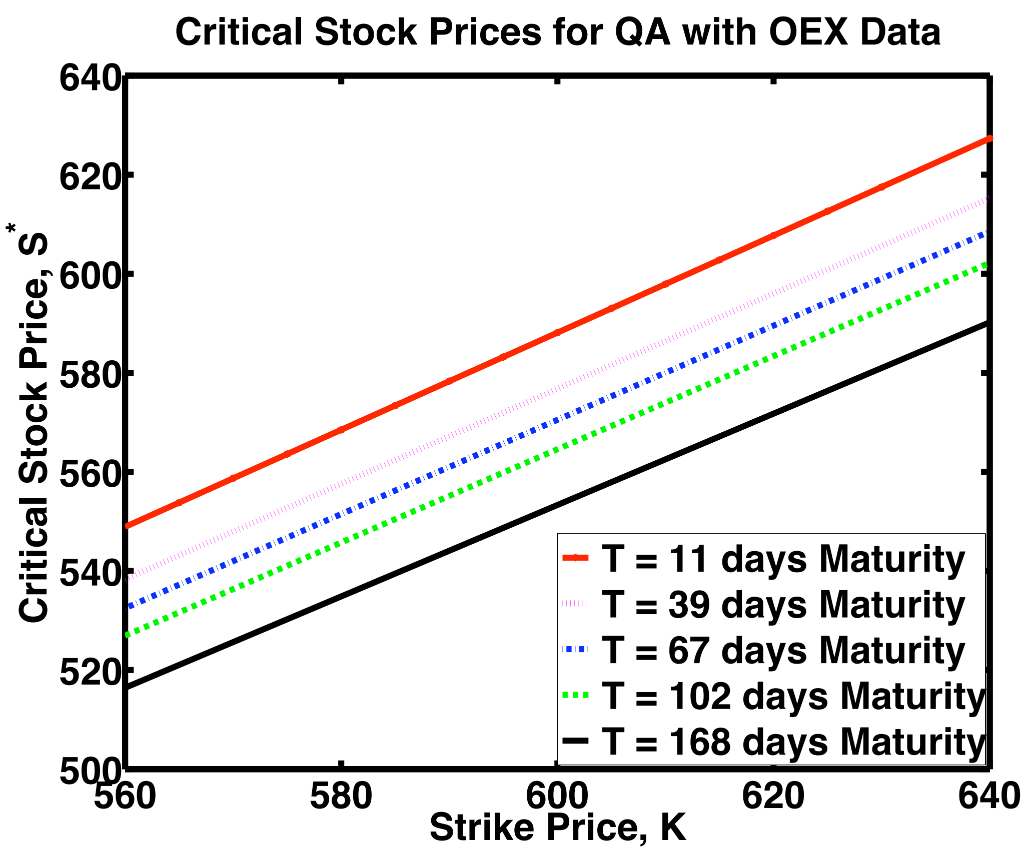


Figure 10.6: **Critical stock prices S^* using QA versus K with OEX quote data**, when $S = \$100$ and $V = 0.01$ ($\sqrt{V} = 0.1$) for $T = \{11, 39, 67, 102, 168\}$ days.

$*$ 10.1.8 Conclusions for American Put Options:

- An **alternative stochastic-volatility jump-diffusion (SVJD)** stock model is proposed with square root mean reverting for stochastic-volatility **combined with log-uniform jump amplitudes**.
- The **heuristic *quadratic* approximation (QA)** and the **accurate LCP finite difference** scheme for American put option pricing **are compared**, with QA being good and fast for practical purposes.
- The QA results are also **checked against real market American option pricing data** OEX (with XEO for Euro. price base), yielding reasonable results considering the simpicity of QA.

• 10.2. SVJD Optimal Portfolio and Consumption Problem

∗ 10.2.0. SVJD Optimal Portfolio and Consumption Problem Outline:

1. Introduction.
2. Optimal Portfolio Problem and Underlying SVJD Model.
3. Portfolio Stochastic Dynamic Programming.
4. CRRA Canonical Solution to Optimal Portfolio Problem.
5. Computational Results.
6. Conclusions.

∗ 10.2.1. Introduction to SVJD Extension of Merton Portfolio Optimization Problem:

{Note: Some of the beginning of this part repeats somethings of the 10.1, first part of L10, but that is for completeness.}

- ***Merton pioneered the optimal portfolio and consumption problem*** for geometric diffusions used HARA (hyperbolic absolute risk-aversion) utility in his lifetime portfolio [Merton, RES (1969)] and general portfolio [Merton, JET (1971)] papers. However, there were some errors, in particular with bankruptcy boundary conditions and vanishing consumption.
- The optimal portfolio errors are throughly discussed in the collection of papers of Sethi's bankruptcy book (1997). See Sethi's introduction, [Karatzas et al., MOR (1986)] and [Sethi & Taksar, JET (1988)].

$*$ 10.2.2. Underlying Stochastic-Volatility, Jump-Diffusion (SVJD) Return Model: [*Hanson (BFS2008)*]

$\circ$ *Stock Price Linear Stochastic Differential Equation (SDE):*

$$
\begin{aligned}
dS(t) = {} & S(t)(\mu_s(t)dt + \sqrt{V}(t)dG_s(t)) \\
& + d\mathrm{CP}_s(t, S(t)\nu(Q)),
\end{aligned}
\tag{10.15}
$$

where the compound Poisson jump process is

$$
\mathrm{CP}_s(t, S(t)\nu(Q)) = \sum_{j=1}^{P(t)} S(T_j^-)\nu_s(Q_j),
$$

and

- $S(t)$ = *stock price*, $S(0) = S_0 > 0$;
- $\mu_s(t)$ = ***expected rate of return*** in absence of asset jumps;
- $V(t)$ = ***stochastic variance*** = (stochastic volatility)^2 = "$\sigma_s^2(t)$";

- $G_s(t)$ = ***stock price diffusion process***, normally distributed such that $\mathrm{E}[dG_s(t)] = 0$ and $\mathrm{Var}[dG_s(t)] = dt$, replacing $dW_s(t)$ since wealth process $W(t)$ has priorty for symbol W;
- $P_s(t)$ = ***Poisson jump counting process***, Poisson distributed such that $\mathrm{E}[dP_s(t)] = \lambda_s(t)dt = \mathrm{Var}[dP_s(t)]$;
- $\nu_s(v, t, q)$ = ***Poisson jump-amplitude*** with underlying ***random mark variable*** $q = Q$, selected for log-return so that $Q = \ln(1 + \nu_s(v, t, Q))$, such that $\nu_s(v, t, q) > -1$;
- T_k^- is the ***pre-jump time*** and Q_k is an independent and identically distributed ***(IID) mark*** realization at the kth jump;
- **Processes $dG_s(t)$ and $P_s(t) = P_s(t; Q)$ along with Q_k are independent**, except that Q_k is conditioned on a jump-event at T_k.

◦ Stochastic-Volatility (Square-Root Diffusion) Model:

[CIR, Econometrica (1985); Heston, RFS (1993); FPS, book (2000)]

$$dV(t) = \kappa_v(t)\,(\theta_v(t) - V(t))\,dt + \sigma_v(t)\sqrt{V(t)}dG_v(t), \quad (10.16)$$

with

- $V(t) \geq \min(V(t)) > 0^+$, $V(0) = V_0 \geq \min(V(t)) > 0^+$;
- ***Log-rate*** $\kappa_v(t) > 0$; ***reversion-level*** $\theta_v(t) > 0$; ***volatility of volatility (variance)*** $\sigma_v(t) > 0$;
- $G_v(t) =$ ***variance diffusion process***, normally distributed such that $\mathrm{E}[dG_v(t)] = 0$ and $\mathrm{Var}[dG_v(t)] = dt$, with ***correlation*** $\mathrm{Corr}[dG_s(t), dG_v(t)] = \rho(t)dt$;
- Note: ***SDE (10.16) is singular for transformations*** as $V(t) \to 0^+$ due to the square root, unlike SDE (10.1) for $S(t)$ where the singularity is removable through the log transformation, but Itô-Taylor chain rule or simulation applications might not be valid unless $\Delta t \ll \sqrt{\min(V(t))} \ll 1$.

○ ***Double-Uniform Jump-Amplitude Q Mark Distribution:***

[*Zhu and Hanson, book chapter preprint (Sethi2006)*]

$$\Phi_Q(q;v,t) = p_1(v,t)\frac{q-a(v,t)}{|a|(v,t)}I_{\{a(v,t)\le q<0\}} + \left(p_1(v,t)+p_2(v,t)\frac{q}{b(v,t)}\right)I_{\{0\le q<b(v,t)\}} + I_{\{b(v,t),\le q<\infty\}}, \quad q\in[a(v,t),b(v,t)],$$

where $a(v,t)<0<b(v,t)$, $p_1(v,t)+p_2(v,t)=1$,

- ***Mark Mean:***
  $\mu_j(v,t)\equiv \mathrm{E}_Q[Q]=(p_1(v,t)a(v,t)+p_2(v,t)b(v,t))/2$;
- ***Mark Variance:*** $\sigma_j^2(v,t)\equiv \mathrm{Var}_Q[Q]=$
  $(p_1(v,t)a^2(v,t)+p_2(v,t)b^2(v,t))/3-\mu_j^2(v,t)$;
- ***More motivation: Double-uniform distribution unlinks the different extreme behaviors in crashes and rallies.***

◦ *Wealth Portfolio with Bond, Stock and Consumption:*

- ***Portfolio:*** Riskless asset or ***bond*** at price $B(t)$ and Risky asset or ***stock*** at price $S(t)$ (10.15), with ***instantaneous*** portfolio change fractions $U_b(t)$ and $U_s(t)$, respectively, such that $U_b(t) = 1 - U_s(t)$.
- ***Exponential Bond Price Process:***

$$dB(t) = r(t)B(t)dt, \quad B(0) = B_0.$$

- **SVJD Portfolio Wealth Process $W(t)$, Less Consumption $C(t)$ with Self-Financing $\{dW/W = (1-U_s)dB/B + U_s dS/S - Cdt/W\}$:**

$$dW(t) = W(t)\Big(r(t)dt + U_s(t)\Big((\mu_s(t) - r(t))dt \\ + \sqrt{V(t)}dG_s(t)\Big)\Big) + d\mathrm{CP}_s(t, W(t)U_s(t)\nu(Q)) - C(t)dt, \tag{10.17}$$

subject to constraints $W(0) = W_0 > 0$, $W(t) > 0$, $v = V(t) > 0$, $0 < C(t) \le C_0^{(\max)}(v,t)W(t)$ and $U_0^{(\min)}(v,t) \le U_s(t) \le U_0^{(\max)}(v,t)$, while allowing extra shortselling ($U_s(t) < 0$) and extra borrowing ($U_b(t) < 0$).

$*$ 10.2.3. SVJD Portfolio Optimal Objective — The Maximal, Expected Utilities of Final Wealth and Running Consumption:

$$
\begin{aligned}
e^{-\overline{\beta}(t)} J^*(w,v,t) = \max_{\{u,c\}} \Bigg[\mathrm{E} \Bigg[& e^{-\overline{\beta}(t_f)} \mathcal{U}_w(W(t_f)) \\
& + \int_t^{t_f} e^{-\overline{\beta}(\tau)} \mathcal{U}_c(C(\tau)) d\tau \\
& \Bigg| W(t)=w, V(t)=v, U_s(t)=u, C(t)=c \Bigg] \Bigg] .
\end{aligned}
\tag{10.18}
$$

where

- ***Cumulative Discount back to $t = 0$:*** $\overline{\beta}(t) = \int_0^t \beta(\tau) d\tau$, where $\beta(t)$ is the **instantaneous discount rate**. The $t_f = T$ is the **final time**.
- ***Consumption and Final Wealth Utility Functions:*** $\mathcal{U}_c(c)$ and $\mathcal{U}_w(w)$ are bounded, strictly increasing and concave.
- ***Variable Classes:*** **State variables** are w and v, while **control variables** are u and c.
- ***Final Condition:*** $J^*(w,v,t_f) = \mathcal{U}_w(w)$.

∘ *Absorbing Natural Boundary Condition:*

Approaching bankruptcy as $w \longrightarrow 0^+$, then, by the consumption constraint, as $c \longrightarrow 0^+$ and by the objective (10.18),

$$e^{-\overline{\beta}(t)} J^*\left(0^+, v, t\right) = \mathcal{U}_w\left(0^+\right) e^{-\overline{\beta}(t_f)} + \mathcal{U}_c\left(0^+\right) \int_t^{t_f} e^{-\overline{\beta}(s)} ds. \quad (10.19)$$

- This is the simple variant what Merton gave as a correction in his 1990 book for his 1971 optimal portfolio paper.
- However, [Karatzas, Lehoczky, Sethi and Shreve (KLASS) (1986) and [Sethi and Taksar (1988)] pointed out that it was necessary to enforce the non-negativity of wealth and consumption.

○ *Derivation of Stochastic Dynamic Programming PIDE by Stochastic Calculus:*

Assume that the optimization and expectation of state and control stochastic processes can be decomposed into independent increments over nonoverlapping time intervals by **Bellman's Principle of Optimality** [Hanson (2007), Ch. 6 & Ex. 6.3] , so that

$$
\begin{aligned}
e^{-\overline{\beta}(t)}J^*(w,v,t) = \max_{\{U,C\}(t,t+\Delta t]} \Bigg[\mathrm{E}_{\{G,CP_Q\}(t,t+\Delta t]} \Bigg[& \int_t^{t+\Delta t} e^{-\overline{\beta}(\tau)}\mathcal{U}_c(C(\tau))d\tau \\
& + e^{-(\overline{\beta}+\Delta\overline{\beta})(t)} \\
& \cdot J^*((W+\Delta W)(t),(V+\Delta V)(t),t+\Delta t) \\
& \Bigg| W(t)=w, V(t)=v, U(t)=u, C(t)=c \Bigg]\Bigg].
\end{aligned}
\tag{10.20}
$$

Next, the limit is taken using the stochastic calculus.

As $\Delta t \to 0^+$, we simplify the state SΔE notation as $\Delta W \stackrel{\Delta t}{=} \mu_w \Delta t + \sigma_w \Delta G_s + \nu_w \Delta P_s$, $\nu_w = uw(\exp(Q)-1)$ and $\Delta V \stackrel{\Delta t}{=} \mu_v \Delta t + \widehat{\sigma}_v \Delta G_v$, while using $J^* = J^*(w,v,t)$ and conditional values, so

$$
\begin{aligned}
e^{-\overline{\beta}(t)} J^*(w,v,t) \stackrel{\Delta t}{=} \max_{\{u,c\}} \Big[e^{-\overline{\beta}(t)} \Big(\mathcal{U}_c(c)\Delta t + J^* + \Delta t \Big(-\beta(t) J^* \\
+ J_t^* + J_w^* \mu_w + J_v^* \mu_v \\
+ 0.5 J_{ww}^* \sigma_w^2 + \rho_w \sigma_w \widehat{\sigma}_v J_{wv}^* + 0.5 J_{vv}^* \widehat{\sigma}_v^2 \\
+ \lambda_s \int_{\mathcal{Q}} dq \phi_Q(q) \\
\cdot (J^*(w + u(e^q - 1)w, v, t) - J^*(w,v,t)) \Big) \Big) \Big].
\end{aligned}
\tag{10.21}
$$

Cancellation of $e^{-\overline{\beta}(t)} J^*(w,v,t)$ on both sides and Δt, yields

$$
\begin{aligned}
0 = \max_{\{u,c\}} \Big[\mathcal{U}_c(c) - \beta(t) J^* + J_t^* + J_w^* \mu_w + J_v^* \mu_v \\
+ 0.5 J_{ww}^* \sigma_w^2 + \rho_w \sigma_w \widehat{\sigma}_v J_{wv}^* + 0.5 J_{vv}^* \widehat{\sigma}_v^2 \\
+ \lambda_s(t) \int_{\mathcal{Q}} dq \phi_Q(q) (J^*(w + u(e^q - 1)w, v, t) - J^*(w,v,t)) \Big].
\end{aligned}
\tag{10.22}
$$

Next, we subtitute for temporary coefficients and take the maximum $(^*)$.

$*$ 10.2.4. SVJD Portfolio Stochastic Dynamic Programming PIDE for Double-Uniform Qs:

$$
\begin{aligned}
0 = & J^*_t(w,v,t) - \beta(t)J^*(w,v,t) + \mathcal{U}_c(c^*) - c^* J^*_w(w,v,t) \\
& + \left(r(t) + (\mu_s(t) - r(t))u^*\right) w J^*_w(w,v,t) \\
& + \kappa_v(t)(\theta_v(t) - v)J^*_v(w,v,t) + \tfrac{1}{2}v(u^*)^2 w^2 J^*_{ww}(w,v,t) \\
& + \tfrac{1}{2}\sigma^2_v(t) v J^*_{vv}(w,v,t) + \rho_v(t)\sigma_v(t) v u^* w J^*_{wv}(w,v,t) \\
& + \lambda_s(t)\left(\frac{p_1(v,t)}{|a|(v,t)}\int_{a(v,t)}^{0} + \frac{p_2(v,t)}{b(v,t)}\int_0^{b(v,t)}\right) \\
& \cdot \left(J^*(\underline{(1 + (e^q - 1)u^*)w}, v, t) - J^*(w,v,t)\right) dq,
\end{aligned}
\tag{10.23}
$$

where $u^* = u^*(w,v,t) \in \left[U_0^{(\mathrm{min})}(v,t), U_0^{(\mathrm{max})}(v,t)\right]$ and $c^* = c^*(w,v,t) \in \left[0, C_0^{(\mathrm{max})}(v,t)w\right]$ are the optimal controls, if they exist, while $J^*_w(w,v,t)$ and $J^*_{ww}(w,v,t)$ are the continuous partial derivatives with respect to wealth w. Note that $\underline{(1 + (e^q - 1)u^*(w,v,t))w}$ **is a post-jump wealth argument.**

$*$ 10.2.5. Positivity of Wealth with Jump Distribution:

Since $(1+(e^q-1)u^*(w,v,t))w$ is a wealth argument in (10.23), it must satisfy the wealth positivity condition, so

$$K(u,q) \equiv 1+(e^q-1)u > 0$$

on $[a(v,t),b(v,t)]$ of the jump-amplitude density $\phi_Q(q;v,t)$.

Lemma 10.1 ***Bounds on Optimal Stock Fraction due to Positivity of Wealth Jump Argument:***

(a) If the support of $\phi_Q(q;v,t)$ is the ***finite*** interval $q \in [a(v,t),b(v,t)]$ with $a(v,t)<0<b(v,t)$, then $u^*(w,v,t)$ is restricted by (10.23) to

$$\frac{-1}{\nu_s(v,t,b(v,t))} < u^*(w,v,t) < \frac{-1}{\nu_s(v,t,a(v,t))}, \tag{10.24}$$

where $\nu_s(v,t,q) = \exp(q)-1$.

(b) If the support of $\phi_Q(q;v,t)$ is fully ***infinite***, i.e., $(-\infty,+\infty)$, then $u^*(w,v,t)$ is restricted by (10.23) to

$$0 < u^*(w,v,t) < 1. \tag{10.25}$$

◦ ***Remarks: Non-Negativity of Wealth and Jump Distribution:***

- Recall that u is the stock fraction, so that ***short-selling and borrowing will be overly restricted in the infinite support case (10.25)*** where $a(v,t) \to -\infty$ and $b(v,t) \to +\infty$, ***unlike the finite case (10.24)***, where $-\infty < a(v,t) < 0 < b(v,t) < +\infty$.
- So, unlike option pricing, ***finite support of the mark density makes a big difference*** in the optimal portfolio and consumption problem!
- Thus, it would ***not be practical to use either normally or double-exponentially distributed marks in the optimal portfolio and consumption problem*** with a bankruptcy condition.
- If $[a_{\min}, b_{\max}] = [\min_t(a(v,t)), \max_t(b(v,t))]$, then the overall u^* range for the S&P500 data used is

$$[u_{\min}, u_{\max}] = [-18, +12] \subset \left(\frac{-1}{(e^{b_{\max}} - 1)}, \frac{+1}{(1 - e^{a_{\min}})} \right).$$

- **Extreme tail ordering:**

$$\exp(-x^2) \ll |x|^N \exp(-|x|) \ll |x|^{-N} \ll 1,\ |x| \gg 1,\ N > 0.$$

$*$ 10.2.6. Unconstrained Optimal or Regular Control Policies:

In absence of control constraints and in presence of sufficient differentiability, the dual policy, implicit critical conditions are

- ***Regular Consumption*** $c^{(\mathrm{reg})}(w,v,t)$ ***{Implicitly}:***

$$\mathcal{U}'_c(c^{(\mathrm{reg})}(w,v,t)) = J^*_w(w,v,t). \tag{10.26}$$

- ***Regular Portfolio Fraction*** $u^{(\mathrm{reg})}(w,v,t)$ ***{Implicitly}:***

$$\begin{aligned} vw^2 J^*_{ww}(w,v,t)u^{(\mathrm{reg})}(w,v,t) &= -(\mu_s(t)-r(t))wJ^*_w(w,v,t) \\ &\quad -\rho\sigma_v(t)vwJ^*_{wv}(w,v,t) \\ &\quad -\lambda(t)w\left(\frac{p_1(v,t)}{|a|(v,t)}\int_{a(v,t)}^{0}+\frac{p_2(v,t)}{b(v,t)}\int_{0}^{b(v,t)}\right) \\ &\quad \cdot(e^q-1)J^*_w\left(t,K\left(u^{(\mathrm{reg})}(w,v,t),q\right)w\right)dq. \end{aligned} \tag{10.27}$$

* 10.2.6. CRRA Utilities Canonical Solution to Optimal Portfolio Problem:

- ***Constant Relative Risk-Aversion (CRRA⊂HARA) Power Utilities:***

$$\mathcal{U}_c(x)=\mathcal{U}(x)=\mathcal{U}_w(x)=\left\{\begin{array}{l} x^\gamma/\gamma,\ \ \gamma\neq 0 \\ \ln(x),\ \ \gamma=0 \end{array}\right\},\ x\geq 0, \gamma<1. \quad (10.28)$$

- ***$\Longleftarrow$ Relative Risk-Aversion (RRA):***

$$RRA(x)\equiv -\mathcal{U}''(x)/(\mathcal{U}'(x)/x)=(1-\gamma)>0,\ \ \gamma<1,$$

  i.e., negative of ratio of marginal to average change in marginal utilility ($\mathcal{U}'(x)>0$ & $\mathcal{U}''(x)<0$) is a constant.

- ***CRRA Canonical Separation of Variables:***

$$J^*(w,v,t)=\mathcal{U}(w)J_0(v,t),\quad J_0(v,t_f)=1, \quad (10.29)$$

  i.e., if valid, then wealth state dependence is known and only the time-variance dependent factor $J_0(v,t)$ need be determined.

$*$ 10.2.6. Canonical Simplifications with CRRA Utilities:

- ***Regular Consumption Control is Linear in Wealth:***

$$c^{(\mathrm{reg})}(w,v,t) = w\cdot c_0^{(\mathrm{reg})}(v,t) \equiv w/J_0^{1/(1-\gamma)}(v,t), \quad (10.30)$$

where $c_0^{(\mathrm{reg})}(v,t)$ is a wealth fraction, with optimal consumption,

$$c_0^*(v,t) = \max\left[\min\left[c_0^{(\mathrm{reg})}(v,t), C_0^{(\max)}(v,t)\right], 0\right]$$

per w.

- ***Regular Fraction Control is Independent of Wealth:***

$$u^{(\mathrm{reg})}(w,v,t) \equiv u_0^{(\mathrm{reg})}(v,t) = \frac{1}{(1-\gamma)v}\Bigg(\mu_s(t) - r(t) + \frac{\rho\sigma_v(t)vJ_{0,v}(v,t)}{J_0(v,t)} + \lambda_s(t)I_1\left(u_0^{(\mathrm{reg})}(v,t),v,t\right)\Bigg), \quad (10.31)$$

where $v > 0$, in fixed point form, where

$$u^* = u_0^*(v,t) = \max\left[\min\left[u_0^{(\mathrm{reg})}(v,t), U_0^{(\max)}\right], U_0^{(\max)}\right],$$

and

$$I_1(u,v,t) \equiv \left(\frac{p_1(v,t)}{|a|(v,t)}\int_{a(v,t)}^{0} + \frac{p_2(v,t)}{b(v,t)}\int_0^{b(v,t)}\right)(e^q - 1)K^{\gamma-1}(u,q)dq.$$

* 10.2.7. CRRA Time-Variance Dependent Component in Formal "Bernoulli" PDE ($\gamma \neq 0; \gamma < 1$):

$$
\begin{aligned}
0 = & \ J_{0,t}(v,t) + (1-\gamma)\left(g_1 J_0 + g_2 J_0^{\frac{\gamma}{\gamma-1}}\right)(v,t) \\
& + g_3(v,t) J_{0,v}(v,t) + \tfrac{1}{2}\sigma_v^2(t) v J_{0,vv}(v,t),
\end{aligned}
\tag{10.32}
$$

where

- ***Bernoulli Coefficients*** $g_1(v,t)$, $g_2(v,t)$, ***and*** $g_3(v,t)$:
  $g_1(v,t) = g_1(v,t;u_0^*(v,t))$,
  $g_2(v,t) = g_2\left(v,t;c_0^*(v,t),c_0^{(\mathrm{reg})}(v,t)\right)$, and $g_3(v,t) = g_3(v,t;u_0^*(v,t))$, introduce implicit nonlinear dependence on $u_0^*(v,t)$, $c_0^*(v,t)$ and $c_0^{(\mathrm{reg})}(v,t)$, so iterations are required.
- ***Formal (Implicit) Solution using Bernoulli transformation***, $J_0(v,t) = y^{1-\gamma}(v,t)$, improving interations:

$$
0 = y_t(v,t) + g_1(v,t) y(v,t) + g_4(v,t), \quad y(v,t_f) = 1,
$$

$$
J_0(v,t) = \left[e^{\overline{g}_1(v,t,tf)} + \int_t^{t_f} g_4(v,\tau) e^{\overline{g}_1(v,t,\tau)} d\tau\right]^{1-\gamma}. \tag{10.33}
$$

Here,

$$g_1(v,t) \equiv \frac{1}{1-\gamma}\Big(-\beta(t)+\gamma\left(r(t)+(\mu_s(t)-r(t))u_0^*(v,t)\right)$$
$$-\frac{1}{2}(1-\gamma)v(u_0^*)^2(v,t)+\lambda_s(t)\left(I_2(u_0^*(v,t),v,t)-1\right)\Big)\,,$$

$$\overline{g}_1(v,t,\tau) \equiv \int_t^\tau g_1(v,s)ds.$$

$$I_2(u,v,t) \equiv \left(\frac{p_1(v,t)}{|a|(v,t)}\int_{a(v,t)}^0 + \frac{p_2(v,t)}{b(v,t)}\int_0^{b(v,t)}\right)K^\gamma(u,q)dq,$$

$$g_2(v,t) \equiv \frac{1}{1-\gamma}\left(\left(\frac{c_0^*(v,t)}{c_0^{(\mathrm{reg})}(v,t)}\right)^\gamma - \gamma\left(\frac{c_0^*(v,t)}{c_0^{(\mathrm{reg})}(v,t)}\right)\right),$$

$$g_3(v,t) = +\kappa_v(t)(\theta_v(t)-v)+\gamma\rho\sigma_v(t)vu_0^*(v,t),$$

$$g_4(v,t) = g_2(v,t)+g_3(v,t)y_v(v,t)+\frac{1}{2}\sigma_v^2(t)v\left(y_{vv}-\gamma((y_v)^2/y)\right)(v,t).$$

$*$ 10.2.8. CRRA Time-Variance Dependent Component in Formal "Bernoulli" PDE ($\gamma = 0$; Kelly Criterion):

Famous Users: Ed Thorp, Warren Buffet, George Soros.

In this medium risk-averse case of the logarithmic CRRA utility, the formal, implicit canonical solution has two terms,

$$J^*(w,v,t) = \ln(w) J_0(v,t) + J_1(v,t), \tag{10.34}$$

with final boundary conditions $J_0(v,t) = 1$ and $J_1(v,t) = 0$.

The regular controls satisfy,

$$c^{(\mathrm{reg})}(w,v,t) = w c_0^{(\mathrm{reg})}(v,t) \equiv w / J_0(v,t),$$

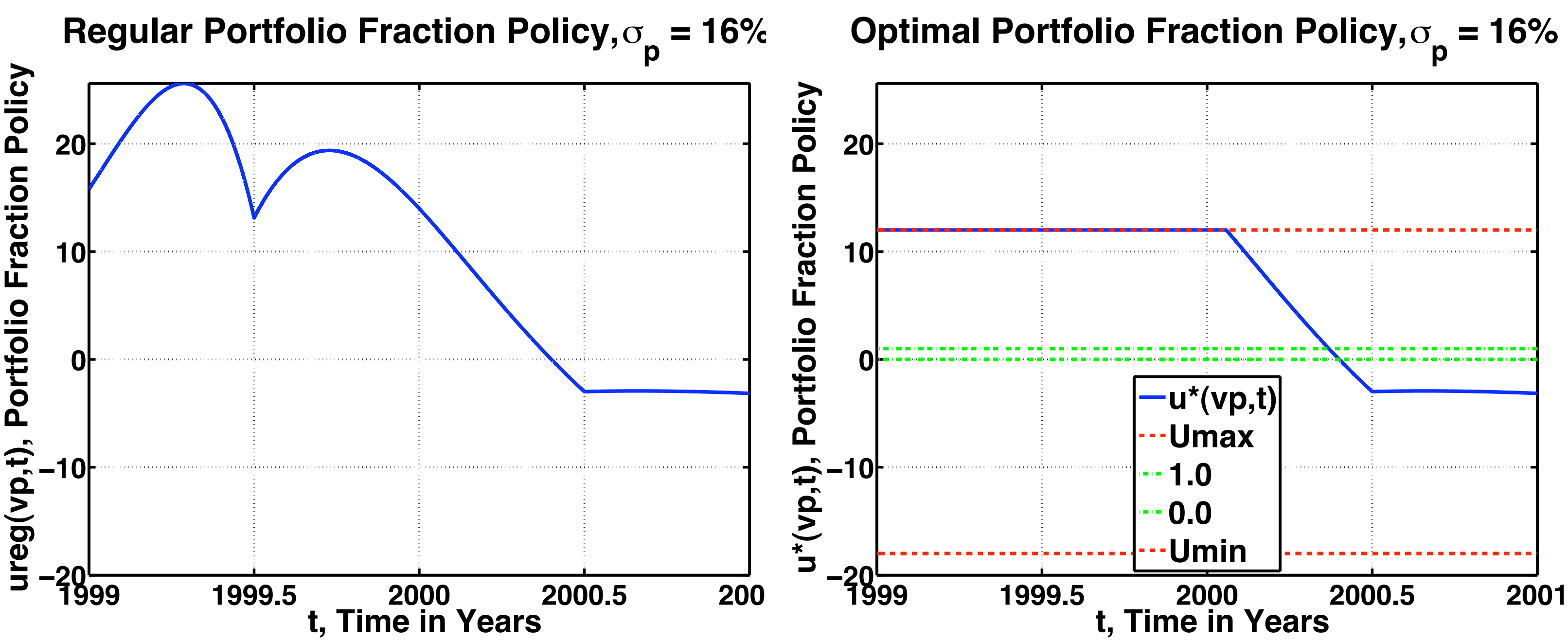
$$\begin{aligned} u^{(\mathrm{reg})}(w,v,t) &= u_0^{(\mathrm{reg})}(v,t) \\ &\equiv \frac{1}{v}\Big(\mu_s(t) - r(t) + \rho\sigma_v(t)(J_{0,v}/J_0)(v,t) \\ &\quad + \lambda_s(t) I_1\Big(u_0^{(\mathrm{reg})}(v,t), v, t\Big)\Big), \quad v > 0. \end{aligned}$$

$\ast$ 10.2.9. Computational Considerations and Results:

◦ *Computational Considerations:*

- The primary problem is having stable computations and much smaller time-steps Δt are needed compared to variance-steps ΔV, since the computations are ***drift-dominated*** over the diffusion coefficient, in that the mesh coefficient associated with $J_{0,v}$ can be hundreds times larger than that associated with $J_{0,vv}$ for the variance-diffusion.
- ***Drift-upwinding*** is needed so the finite differences for the drift-partial derivatives follow the sign of the drift-coefficient, while central differences are sufficient for the diffusion partials.
- ***Iteration calculations in time, controls and volatility*** are sensitive to small and negative deviations, as well as the form of the iteration in terms of the formal implicitly-defined solutions.

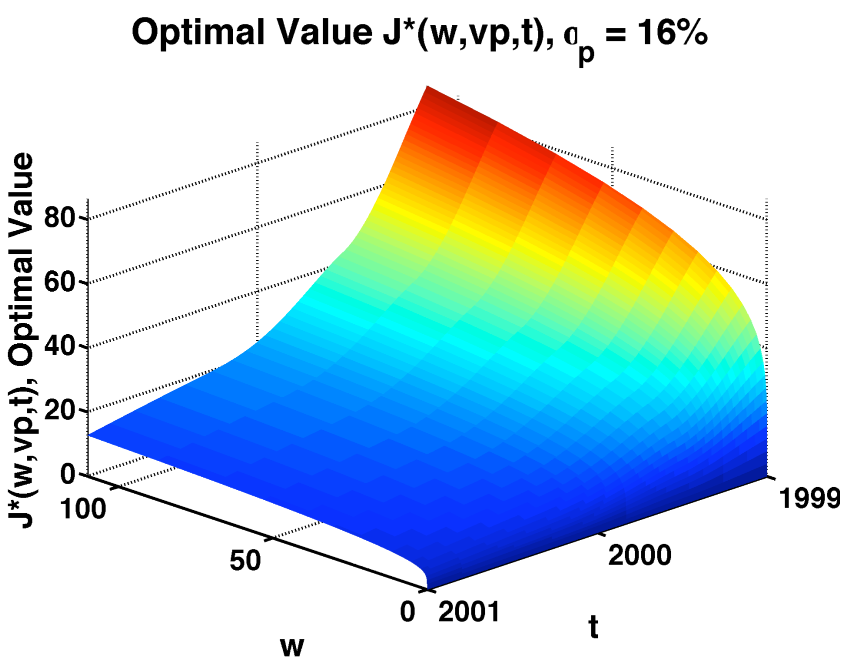
○ ***Results for Regular $u^{(reg)}(v_p, t)$ and Optimal $u^*(v_p, t)$ Portfolio Fraction Policies, $\sigma_p = \sqrt{v_p} = 16\%$:***



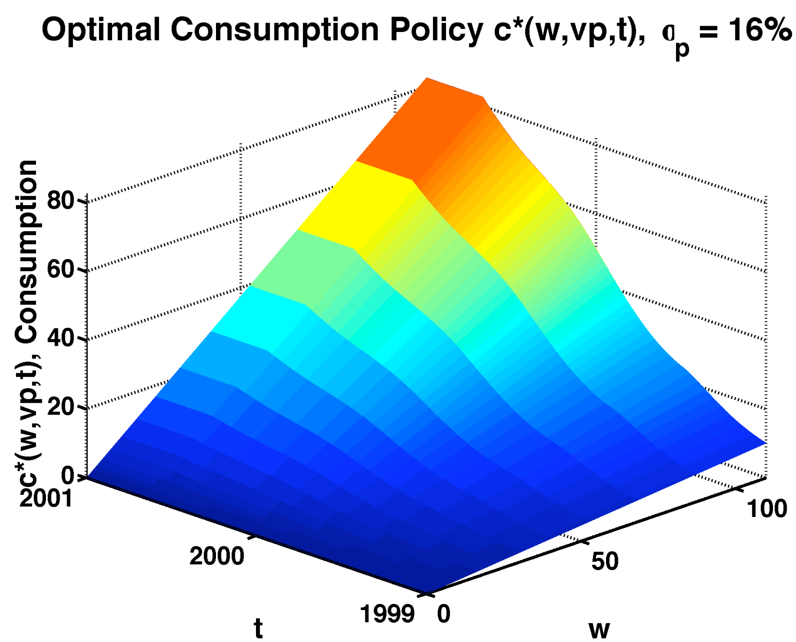
(a) Regular fraction policy $u^{(reg)}(v_p, t)$. (b) Optimal fraction policy, $u^*(v_p, t)$.

Figure 10.7: Regular and optimal portfolio stock fraction policies, $u^{(reg)}(v_p, t)$ and $u^*(v_p, t)$ on $t \in [1999.0, 2001.0]$, while $u^*(v_p, t) \in [-18, 12]$.

○ ***Results for Optimal Value*** $J^*(w, v_p, t)$ ***and Optimal Consumption*** $c^*(w, v_p, t)$***, Portfolio Fraction Policies,*** $\sigma_p = \sqrt{v_p} = 16\%$ ***:***



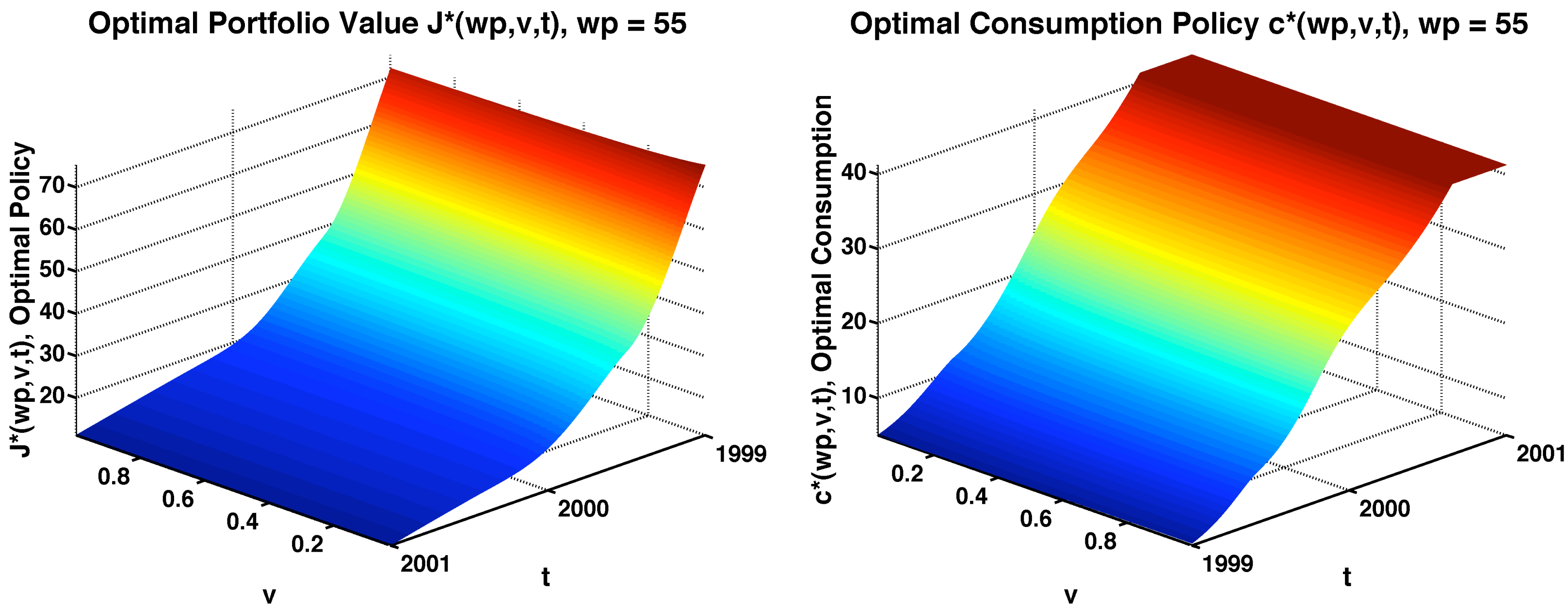
(a) Optimal portfolio value $J^*(w, v_p, t)$.



(b) Optimal consumption policy $c^*(w, v_p, t)$.

Figure 10.8: Optimal portfolio value $J^*(w, v_p, t)$ and optimal consumption policy $c^*(w, v_p, t)$ for $(w, v_p, t) \in [0, 110] \times [1999.0, 2001.0]$, while $c^*(w, v_p, t) \in [0, 0.75 \cdot w]$ is enforced near $t = 2001$.

- ***Results for Optimal Value $J^*(w_p, v, t)$ and Optimal Consumption $c^*(w_p, v, t)$, $w_p = 55$:***



(a) Optimal portfolio value $J^*(w_p, v, t)$. (b) Optimal consumption $c^*(w_p, v, t)$.

Figure 10.9: Optimal portfolio value $J^*(w_p, v, t)$ and optimal consumption $c^*(w_p, v, t)$ at $w_p = 55$ for $(v, t) \in \times[v_{\min}, 1.0] \times [1999.0, 2001.0]$, while $c^*(w_p, v, t) \in [0, 0.75 \cdot w_p]$ is enforced near $t = 2001$.

- ***Results for Optimal Portfolio Fraction $u^*(v,t)$:***

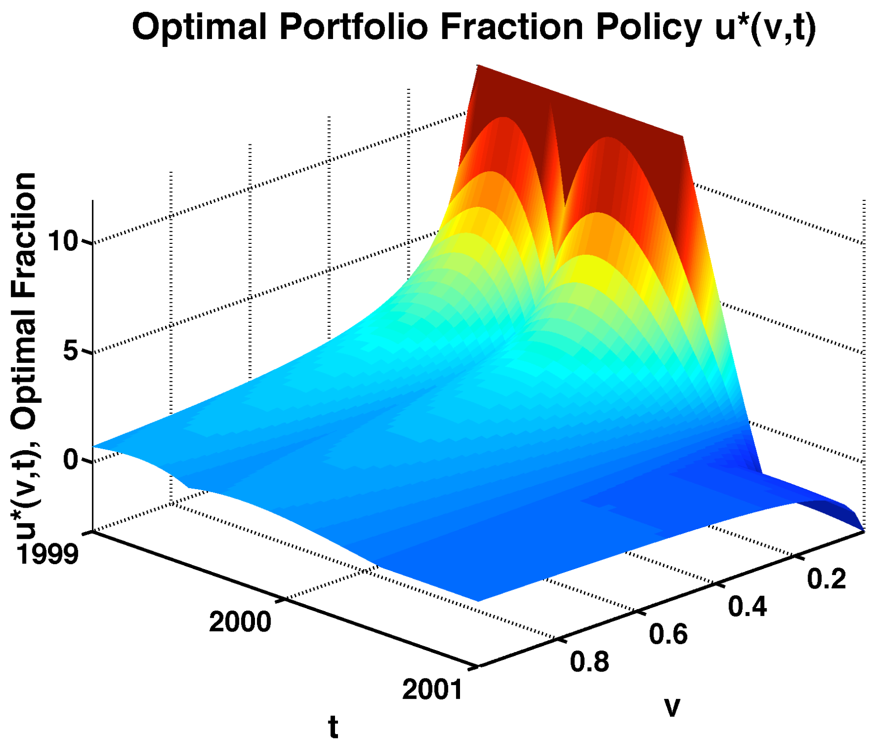


Figure 10.10: Optimal portfolio fraction policy $u^*(v,t)$ for $(v,t) \in \times[v_{\min}, 1.0] \times [1999.0, 2001.0]$, while $u^*(v,t) \in [-18, 12]$ is enforced near small variance $v = v_{\min} > 0$.

$*$ 10.2.10. Conclusions for SVJD Optimal Portfolio and Consumption Problem :

- ***Generalized the optimal portfolio and consumption problem for jump-diffusions to include stochastic volatility/variance*** .
- Confirmed significant effects on ***variation of instantaneous stock fraction policies*** due to time-dependence of interest and discount rates for SVJD optimal portfolio and consumption models.
- ***Showed jump-amplitude distributions with compact support are much less restricted on short-selling and borrowing*** compared to the infinite support case in the SVJD optimal portfolio and consumption problem.
- Noted that the CRRA reduced canonical optimal portfolio problem is ***strongly drift-dominated*** for sample market parameter values over the diffusion terms, so at least first order drift-upwinding is essential for stable Bernoulli PDE computations.

Summary of Lecture 10?

1.

2.

3.

4.

5.