FinM 345/Stat 390 Stochastic Calculus, Autumn 2009

Floyd B. Hanson, Visiting Professor

Email: fhanson@uchicago.edu

Master of Science in Financial Mathematics Program
University of Chicago

Lecture 4 (from Singapore)
Jump & Jump-Diffusion Stochastic Calculus

6:30-9:30 pm, 19 October 2009, Kent 120 in Chicago

7:30-10:30 pm, 19 October 2009 at UBS Stamford

7:30-10:30 am, 20 October 2009 at Spring in Singapore

FinM 345 Stochastic Calculus:

4. Jump & Jump-Diffusion Stochastic Calculus:

4.1. Poisson Jump Calculus Basic Chain Rules:

The Poisson process is quite different from the continuous diffusion process, primarily because of the discontinuity property of the Poisson process and the property that multiple jumps are highly unlikely during small increments in time Δt.

- **4.1.1. Jump Calculus Rule for $h(dP(t))$**: Thus, the most basic rule is the *zero-one law (ZOL) for jumps for* $dP(t)$ in precision-dt compact differential form,

$$(dP)^m(t) \underset{\text{zol}}{\overset{\text{dt}}{=}} dP(t), \tag{4.1}$$

provided the integer $m \geq 1$, the case $m = 0$ being trivial.

An immediate generalization of this law is the following corollary.

Corollary 4.1 Zero-One Jump Law for $h(dP(t))$:

$$h(dP(t)) \overset{\text{dt}}{\underset{\text{zol}}{=}} h(1)dP(t) + h(0)(1 - dP(t)), \quad (4.2)$$

with probability one, provided the function $h(p)$ is right-continuous, such that values $h(0)$ and $h(1)$ exist and are bounded.

Proof: This follows by simple substitution of the *zero-one jump law*,

$$h(dP(t)) \overset{\text{dt}}{\underset{\text{zol}}{=}} \left\{ \begin{array}{l} h(1),\ dP(t)=1 \\ h(0),\ dP(t)=0 \end{array} \right\} \overset{\text{dt}}{\underset{\text{zol}}{=}} h(1)dP(t)+h(0)(1-dP(t)),$$

$dP(t) = 0$ or $dP(t) = 1$ with probability one to precision-dt. □

Formally, the differential $dP(t)$ can be treated as a **condition to test whether there has been a jump**. This form (4.2) of the zero-one law suggests another extension of the **jump function definitions** (B.178-B.179). For example, recall in (B.185) for a jump at t_1,

$$[F](X(t_1), t_1) = F(X(t_1^+), t_1^+) - F(X(t_1^-), t_1^-).$$

Definition 4.1. Jump Function $[h](dP(t))$:

$$[h](dP(t)) \overset{\text{dt}}{\underset{\text{zol}}{=}} h(dP(t)) - h(0) \tag{4.3}$$

to precision-dt, provided $h(p)$ is right-continuous, such that values $h(0)$ and $h(dP(t))$ exist and are bounded.

With this definition, version (4.2) of the **zero-one law** can immediately be written.

Corollary 4.2 Zero-One Jump Law for $h(dP(t))$ with Jump Function:

$$h(dP(t)) \overset{\text{dt}}{\underset{\text{zol}}{=}} h(0) + [h](dP(t)) \tag{4.4}$$

in terms of the jump function $[h](dP(t))$. Alternatively, the jump function is written as

$$[h](dP(t)) \overset{\text{dt}}{\underset{\text{zol}}{=}} (h(1) - h(0))dP(t). \tag{4.5}$$

- **4.1.2. Jump Calculus Rule for $\mathcal{H}(P(t), t)$:**

  Equations (4.4 – 4.5) are a primitive differential chain rule for functions of only the Poisson differential $dP(t)$. However, more complex rules will be needed, for instance, a chain rule for a combination of a simple Poisson jump process in $P(t)$ and a deterministic process with explicit dependence on t.

  Rule 4.1. Chain Rule for $\mathcal{H}(P(t), t)$:

  Let $\mathcal{H}(p, t)$ be once continuously differentiable in t and right-continuous in p.

$$d\mathcal{H}(P(t), t) \overset{\text{dt}}{\underset{\text{zol}}{=}} \mathcal{H}_t(P(t), t)dt + [\mathcal{H}](P(t), t), \quad (4.6)$$

where

$$[\mathcal{H}](P(t),t) \stackrel{\text{dt}}{\underset{\text{zol}}{=}} (\mathcal{H}(P(t)+1,t) - \mathcal{H}(P(t),t))\, dP(t) \quad (4.7)$$

is the corresponding jump function definition for functions of $P(t)$ and t.

Begin Sketch of Proof: Proceeding formally with differential precision-dt, the differential definition as an increment yields

$$\begin{aligned} d\mathcal{H}(P(t),t) &= \mathcal{H}(P(t+dt),t+dt) - \mathcal{H}(P(t),t) \\ &= \mathcal{H}(P(t)+dP(t),t+dt) - \mathcal{H}(P(t),t). \end{aligned}$$

Next, using the zero-one jump law (4.2) for $h(dP(t))$ on $\mathcal{H}(P(t)+dP(t),t+dt)$ for fixed $(P(t),t)$ to take $dP(t)$ out of its first argument and then expanding the second argument dt to two terms up to $\mathcal{H}_t$,

$$
\begin{aligned}
d\mathcal{H}(P(t),t) &\overset{\mathrm{dt}}{\underset{\mathrm{zol}}{=}} \mathcal{H}(P(t)+1,t+dt)dP(t) \\
&\quad +\mathcal{H}(P(t)+0,t+dt)(1-dP(t))-\mathcal{H}(P(t),t) \\
&\overset{\mathrm{dt}}{\underset{\mathrm{zol}}{=}} (\mathcal{H}(P(t),t)+\mathcal{H}_t(P(t),t)dt)(1-dP(t)) \\
&\quad +(\mathcal{H}(P(t)+1,t)+\mathcal{H}_t(P(t)+1,t)dt)dP(t) \\
&\quad -\mathcal{H}(P(t),t) \\
&\overset{\mathrm{dt}}{\underset{\mathrm{zol}}{=}} \mathcal{H}_t(P(t),t)dt+(\mathcal{H}(P(t)+1,t) \\
&\quad -\mathcal{H}(P(t),t))dP(t) \\
&\overset{\mathrm{dt}}{\underset{\mathrm{zol}}{=}} \mathcal{H}_t(P(t),t)dt+[\mathcal{H}](P(t),t),
\end{aligned}
$$

the last line due to using the jump function definition (4.7).

Also used was the bilinear differential form

$$dt\, dP(t) \overset{\text{dt}}{\underset{\text{zol}}{=}} 0,$$

which is mainly responsible for the elimination of combined continuous and jump changes.

The precision-dt jump differential **Table 2.2.2** on L2-p51 was used to eliminate terms smaller than precision-dt terms in the mean square sense. The dt factor $\mathcal{H}_t(p,t)$ is the partial derivative of $\mathcal{H}$ with respect to t while p is held fixed. Note that the jump function is defined for all t so that if there is no Poisson jump, then the jump function is identically zero since $dP(t) = 0$, the zero jump case.

□

Remarks:

- The bilinear differential form $dt\,dP(t) \stackrel{\text{dt}}{\underset{\text{zol}}{=}} 0$ is consistent with the fact that the Poisson process has jump discontinuities and thus jumps must be instantaneous.
- **Consequently, continuous changes and jump changes can be computed independently, since there are zero continuous changes at each jump instant.**
- This leads to the alternate form of **Rule 4.1**.

Rule 4.2. Alternate Chain Rule for $\mathcal{H}(P(t), t)$:

Let $\mathcal{H}(p, t)$ be once continuously differentiable in t and right continuous in p.

$$d\mathcal{H}(P(t), t) \overset{\text{dt}}{\underset{\text{zol}}{=}} d_{(\text{cont})}\mathcal{H}(P(t), t) + d_{(\text{jump})}\mathcal{H}(P(t), t), \tag{4.8}$$

where

$$d_{(\text{cont})}\mathcal{H}(P(t), t) \equiv \mathcal{H}_t(P(t), t)dt \tag{4.9}$$

and

$$d_{(\text{jump})}\mathcal{H}(P(t), t) \equiv [\mathcal{H}](P(t), t). \tag{4.10}$$

Example 4.1. Stochastic Jump Power:

Let $a \neq 0$ and $b > 0$. Using the stochastic jump chain rule (4.7) in differential form, we then have

$$d\left[b^{aP(t)+ct}\right] \overset{\text{dt}}{\underset{\text{zol}}{=}} c\ln(b)b^{aP(t)+ct}dt + \left(b^{a(P(t)+1)+ct} - b^{aP(t)+ct}\right)dP(t)$$
$$= b^{aP(t)+ct}(c\ln(b)dt + (b^a - 1)dP(t)),$$

where the calculus rule, $d(b^{ct}) = d(e^{c\ln(b)t}) = c\ln(b)b^{ct}$, for an arbitrary positive power base b with an exponential rule has been used.

The corresponding jump integral derived from this formula is

$$\int_0^t b^{aP(s)+cs}dP(s) \overset{\text{dt}}{=} \frac{1}{b^a - 1}\left(\left(b^{aP(t)+ct} - 1\right) - c\ln(b)\int_0^t b^{aP(s)+cs}ds\right),$$

provided $b^a \neq 1$.

This integral formula simplifies if $b = e$ and $c = 0$ to

$$\int_0^t \exp(aP(s))dP(s) \stackrel{\text{dt}}{=} (\exp(aP(t)) - 1)/(\exp(a) - 1),$$

but still is different from the deterministic version,

$$\int_0^t \exp(as)ds = (\exp(at) - 1)/a.$$

4.1.3. Jump Calculus Rule with General State

$Y(t) = F(X(t), t)$: The chain rule for $F(P(t), t)$ is still too simple, so a chain rule for more general jump processes $X(t)$, such as for $F(X(t), t)$, is needed. First, a definition of a jump function for general transformations is needed.

Definition 4.2. $[Y](t)$ **for General** $Y(t) = F(X(t), t)$:

Let the process $Y(t) = F(X(t), t)$ be a continuous transformation of the process $X(t)$ with jump function $[X](t)$ at t. Then the jump function in $Y(t)$ is defined as

$$\begin{aligned} [Y](t) &= [F](X(t), t) \\ &= F(X(t) + [X](t), t) - F(X(t), t). \end{aligned} \quad (4.11)$$

Lemma 4.1. $[Y](t)$ for $Y(t) = F(X(t), t)$ with $[X](t) = h(X(t), t)dP(t)$:

Let the process $Y(t) = F(X(t), t)$ be a continuous transformation of the process $X(t)$ with jump function

$$[X](t) = h(X(t), t)dP(t)$$

at t, then

$$\begin{aligned} [Y](t) &= [F](X(t), t) \\ &= (F(X(t)+h(X(t), t), t) \\ &\quad -F(X(t), t))\, dP(t). \end{aligned} \tag{4.12}$$

Proof: This follows from the **Zero-One Jump Law** (4.2) for $h(dP(t))$ upon substitution of the jump of $[X](t) = h(X(t),t)dP(t)$ into the definition (4.11), so that

$$\begin{aligned}[Y](t) &\equiv F(X(t)+[X](t),t)-F(X(t),t)\\ &= F(X(t)+h(X(t),t)dP(t),t)-F(X(t),t)\\ &= (F(X(t)+h(X(t),t),t)-F(X(t),t))dP(t).\end{aligned}$$

□

Rule 4.3. Chain Rule for Jump in $Y(t) = F(X(t), t)$:

Let $Y(t) = F(X(t), t)$, such that the function $F(x, t)$ is continuously differentiable once in x and once in t. Let the $X(t)$ process satisfy the jump SDE,

$$dX(t) = f(X(t), t)dt + h(X(t), t)dP(t), \quad (4.13)$$

$X(0) = x_0$ with probability one, while $f(X(t), t)$ and $h(X(t), t)$ satisfy the mean square integrability conditions with the $W(t)$ argument replaced by the $X(t)$ arguments of f and h. In (4.13), the jump in $X(t)$ is $[X](T_k^-) \equiv X(T_k^+) - X(T_k^-) = h(X(T_k^-), T_k^-)$ for each kth jump-time T_k of $P(t)$.

Then

$$dY(t) = dF(X(t),t) \stackrel{\text{dt}}{\underset{\text{zol}}{=}} (F_t + fF_x)(X(t),t)dt + [F](X(t),t), \tag{4.14}$$

where wholesale arguments have been used for the coefficient functions multiplying dt and $dP(t)$, respectively, and where the jump in $Y(t) = F(X(t),t)$ is given in (4.12) of Lemma 4.1.

Sketch of Proof: Formally, a sketch of the proof uses the increment form of the differential

$$\begin{aligned} dY(t) &= Y(t+dt) - Y(t) \\ &= F(X(t+dt), t+dt) - F(X(t), t) \\ &= F(X(t) + dX(t), t+dt) - F(X(t), t). \end{aligned}$$

Next, as for (4.6), (4.8) of the two prior rules, the instantaneous jump changes (terms in $dP(t)$ only, such that $[X](t) = h(X(t), t)dP(t)$) are treated separately from the **continuous and smooth deterministic changes**, i.e., terms in dt only, such that

$$dX^{(\mathrm{det})}(t) \equiv f(X(t), t)dt.$$

Then the mean square approximations are used with their implied precision-dt ,

$$
\begin{aligned}
dY(t) &\underset{\text{zol}}{\overset{\text{dt}}{=}} F_t(X(t),t)dt + F_x(X(t),t)f(X(t),t)dt \\
&\quad + (F(X(t)+[X](t),t) - F(X(t),t)) \\
&\underset{\text{zol}}{\overset{\text{dt}}{=}} (F_t + fF_x)(X(t),t)dt \\
&\quad + (F(X(t)+h(X(t),t)dP(t),t) - F(X(t),t)) \\
&\underset{\text{zol}}{\overset{\text{dt}}{=}} (F_t + fF_x)(X(t),t)dt \\
&\quad + (F(X(t)+h(X(t),t),t) - F(X(t),t))\,dP(t),
\end{aligned}
$$

where the **Zero-One Jump Law** (4.5) has been used to take the $dP(t)$ out of the argument of F and let it multiply the jump change in F in the last line of the above equation. **Note that the jump change has been defined, so that if there is no Poisson jump, then the jump function is zero.** □

4.1.4. Transformations of Linear Jump with Drift SDEs

Consider the jump SDE, linear in the state process $X(t)$, with time-dependent coefficients,

$$dX(t) = X(t)\left(\mu(t)dt + \nu(t)dP(t)\right), \qquad (4.15)$$

where here the initial condition is $X(t_0) = x_0 > 0$ with probability one, $\mu(t)$ is called the **drift** or deterministic coefficient and $\nu(t)$ is called the **jump-amplitude** coefficient of the Poisson jump term. The jump in state is $[X](T_k) = \nu(T_k^-)$ for each jump of $P(t)$, i.e., $[P](T_k) = 1$ for each k. Assume that the rate coefficients, $\mu(t)$ and $\nu(t)$ are bounded, while $\nu(t) > -1$ **(one jump and out condition:** $dP(T_1^-) = 1$ **&** $X(T_1^+) = X(T_1^-) + dX(T_1^-) = X(T_1^-) - X(T_1^-) = 0$ **)**.

In the deterministic and linear diffusion cases, transforming the state variable to its logarithm makes the right-hand side independent of the transformed state variable, so let

$$Y(t) = F(X(t)) \equiv \ln(X(t)).$$

The most recent jump chain rule (4.14), (4.12) is applicable in this case with

$$f(X(t), t) = X(t)\mu(t)$$

and

$$h(X(t), t) = X(t)\nu(t),$$

although the increment form of $dF(X(t))$ can be directly expanded to get the same result.

Since only the first partial derivative and the jump function of F are needed, while F does not depend on t, then

$$F_x(X(t)) = 1/X(t), \;\; F_t(X(t)) \equiv 0,$$

and from (4.12)

$$\begin{aligned}[F](X(t)) &\stackrel{\mathrm{dt}}{\underset{\mathrm{zol}}{=}} (\ln(X(t) + X(t)\nu(t)) \\ &\qquad - \ln(X(t)))\, dP(t) \\ &= \ln(1 + \nu(t))dP(t),\end{aligned} \tag{4.16}$$

where the **logarithm subtraction rule** $\ln(A) - \ln(B) = \ln(A/B)$, provided $A > 0$ and $B > 0$, has been used to cancel out the linear state dependence in the jump term. Note that the jump-amplitude becomes singular and disastrous for $X(t)$ as $\nu(t) \to (-1)^+$.

Thus,

$$dY(t) = dF(X(t))$$
$$= F_x(X(t))X(t)\mu(t)dt + [F](X(t)) \qquad (4.17)$$
$$\underset{\text{zol}}{\overset{\text{dt}}{=}} \mu(t)dt + \ln(1+\nu(t))dP(t).$$

The infinitesimal mean of $Y(t)$, assuming the jump rate is time-dependent $\mathrm{E}[dP(t)] = \lambda(t)dt$ too, is

$$\mathrm{E}[dY(t)] = (\mu(t) + \lambda(t)\ln(1+\nu(t)))\,dt \quad (4.18)$$

and the **infinitesimal variance** is

$$\mathrm{Var}[dY(t)] \overset{\text{dt}}{=} \lambda(t)\ln^{2}(1+\nu(t))dt, \qquad (4.19)$$

noting that the jump-amplitude has a power effect between the infinitesimal expectation and the variance unlike the Poisson infinitesimal property that $\mathrm{Var}[dP(t)] = \mathrm{E}[dP(t)]$.

Since the final right-hand side of (4.17) does not depend on the state $\boldsymbol{Y(t)}$, we can easily integrate for $\boldsymbol{Y(t)}$ explicitly, leading to

$$\boldsymbol{Y(t) = Y(t_0) + \int_{t_0}^{t} \mu(s)ds + \int_{t_0}^{t} \ln(1+\nu(s))dP(s).} \tag{4.20}$$

Exponentiation leads to the formal solution for the original state,

$$\boldsymbol{X(t) = X(t_0)\exp\left(\int_{t_0}^{t} \mu(s)ds + \int_{t_0}^{t} \ln(1+\nu(s))dP(s)\right).} \tag{4.21}$$

• Linear Jump SDEs with Constant Coefficients

If the SDE has **constant coefficients**, $\mu(t) = \mu_0$, $\nu(t) = \nu_0$ and $\lambda(t) = \lambda_0$, then the solution is simpler:

$$\begin{aligned} X(t) &\overset{\text{ims}}{=} X(t_0)\exp\left(\mu_0(t-t_0)\right. \\ &\qquad \left.+\ln(1+\nu_0)(P(t)-P(t_0))\right) \\ &= X(t_0)\exp\left(\mu_0(t-t_0)\right)(1+\nu_0)^{(P(t)-P(t_0))}, \end{aligned} \tag{4.22}$$

where, in the last line, the exponential-logarithm inverse relation, $\exp(a\ln(b)) = b^a$, has been used to move the Poisson term out of the exponential.

In this pure jump with drift process, the moments are computed using the Poisson distribution on p. L1-p42, coupled with the **stationary property that the distribution depends only on the time increment** (and the jump rate),

$$\begin{aligned}\mathrm{Prob}[P(t)-P(t_0)=k] &= \mathrm{Prob}[P(t-t_0)=k] \\ &= p_k(\lambda_0(t-t_0)) \\ &= e^{-\lambda_0(t-t_0)}\frac{(\lambda_0(t-t_0))^k}{k!}.\end{aligned}$$

Thus, the calculation of the mean of the process in (4.22) is

$$\begin{aligned}\mathrm{E}[X(t)] &= x_0 e^{\mu_0(t-t_0)} e^{-\lambda_0(t-t_0)} \sum_{k=0}^{\infty} \frac{(\lambda_0(t-t_0))^k}{k!}(1+\nu_0)^k \\ &= x_0 e^{\mu_0(t-t_0)-\lambda_0(t-t_0)} e^{\lambda_0(t-t_0)(1+\nu_0)} \\ &= x_0 e^{(\mu_0+\lambda_0\nu_0)(t-t_0)},\end{aligned}$$

growing in time if $\mu_0 + \lambda_0\nu_0 > 0$, but decaying if $\mu_0 + \lambda_0\nu_0 < 0$. Note that $\lambda_0 > 0$, but both μ_0 and ν_0 can be of any sign.

The corresponding calculation of the variance of $X(t)$ is

$$\begin{aligned}
\mathrm{Var}[X(t)] &= \mathrm{E}[X^2(t)] - \mathrm{E}^2[X(t)] \\
&= x_0^2 e^{2\mu_0(t-t_0)} e^{-\lambda_0(t-t_0)} \\
&\quad \cdot \sum_{k=0} \frac{(\lambda_0(t-t_0))^k}{k!}(1+\nu_0)^{2k} - \mathrm{E}^2[X(t)] \\
&= x_0^2 e^{2\mu_0(t-t_0)-\lambda_0(t-t_0)} e^{\lambda_0(t-t_0)(1+\nu_0)^2} \\
&\quad - x_0^2 e^{2(\mu_0+\lambda_0\nu_0)(t-t_0)} \\
&= x_0^2 e^{2(\mu_0+\lambda_0\nu_0)(t-t_0)} \left(e^{\lambda_0\nu_0^2(t-t_0)} - 1\right) \\
&= \mathrm{E}^2[X(t)] \left(e^{\lambda_0\nu_0^2(t-t_0)} - 1\right),
\end{aligned}$$

so the growth or decay is proportional to the mean squared, but amplified asymptotically by the growing term $\exp(\lambda_0\nu_0^2(t-t_0))$, as in the diffusion case.

For the distribution, see the forward Subsection 4.2.3 (4.3.3, p. 109 of the textbook) for the linear jump-diffusion SDE case.

Applications include stochastic population growth where $X(t)$ is the population size, such that the population grows exponentially at **intrinsic growth rate** $\mu(t)$ in the absence of stochastic disasters, but suffers from a **random linear disaster** if the jump-amplitude rate $-1 < \nu(t) < 0$ or from a **random linear bonanza** if $\nu(t) > 0$. See also Ryan and Hanson (MB 1985) or Chapter 11 summary on **biological applications**.

4.2. Jump-Diffusion Rules and SDEs:

Wiener diffusion and simple Poisson jump processes provide an introduction to SDEs in continuous time for the **simple jump-diffusion** state process $X(t)$,

$$\begin{aligned} dX(t) = \ & (X(t), t)dt + g(X(t), t)dW(t) \\ & + h(X(t), t)dP(t), \end{aligned} \tag{4.23}$$

where $X(0) = x_0$, with a set of continuous coefficient functions $\{f, g, h\}$, possibly nonlinear in the state $X(t)$. However, in the process of introducing the component Markov processes, too many rules have been accumulated and in this section most of these rules will be combined into one rule or a few rules.

4.2.1. Jump-Diffusion Conditional Infinitesimal Moments:

The conditional infinitesimal moments for the state process are useful for application modeling and are given by

$$\mathrm{E}[dX(t)|X(t)=x]=(f(x,t)+\lambda(t)h(x,t))dt \quad (4.24)$$

and

$$\mathrm{Var}[dX(t)|X(t)=x]=\left(g^2(x,t)+\lambda(t)h^2(x,t)\right)dt, \quad (4.25)$$

using (4.23) and assuming that the Poisson process is independent of the Wiener process.

The jump in the state at jumps T_k in the Poisson process, i.e., $[P](T_k) = 1$, is not an infinitesimal moment but serves as a simple property of the SDE and is given by

$$[X](T_k) \equiv X(T_k^+) - X(T_k^-) = h(X(T_k^-), T_k^-) \quad (4.26)$$

or

$$[X](t) = h(X(t), t)dP(t), \quad (4.27)$$

under the assumptions that the jumps are instantaneous so there are no time-continuous changes for the instant and that in the interval $(t, t + dt]$ there is time for only one jump, if any, of the Poisson term by the zero-one jump rule. Note that no $dP(t)$ appears in (4.26) since a jump is assumed at $t = T_k$.

Infinitesimal moments and jump condition characterize the **simple jump-diffusion**, called simple to distinguish it from the compound Poisson process, in which $h = h(x, t, q)$ denotes the random jump-amplitude form when there is an underlying IID random variable $q = Q$, while h &h^2 need to be replaced by their mean values.

The jump-amplitude evaluation (4.26) at the pre-jump time value T_k^- follows from the Itô forward integration approximation and the right-continuity of $P(t)$, as discussed in the previous chapter and also means that the jump-amplitude depends only on the immediate prejump value of h, but not on the postjump value, which in a sense is in the future.

The infinitesimal moment and jump properties are very useful for modeling approximations of real applications, by providing a basis for estimating the coefficient functions f, g, and h, as well as some of the process parameters, at least in the first approximation, through comparison to the empirical values of the basic probability for the stochastic integral equation.

4.2.2 Stochastic Jump-Diffusion Chain Rule:

The corresponding stochastic chain rule for calculating the differential of a composite process $F(X(t), t)$ begins by interpreting the differential as an infinitesimal increment and recognizing that since the Poisson jumps are instantaneous there is no time for continuous changes. **Thus, a critical concept in deriving the chain rule is that the continuous changes and jump changes can be calculated independently.**

The state process is decomposed into independent continuous changes,

$$d_{(\mathrm{cont})}X(t) = f(X(t),t)dt + g(X(t),t)dW(t) \quad (4.28)$$

and instantaneous discontinuous or jump changes,

$$d_{(\mathrm{jump})}X(t) = [X](t) = h(X(t),t)dP(t), \quad (4.29)$$

such that

$$dX(t) = d_{(\mathrm{cont})}X(t) + d_{(\mathrm{jump})}X(t). \quad (4.30)$$

Another critical concept is the transformation of the conditioning for the jump. The differential Poisson $dP(t)$ serves as the conditioning for the existence of a jump. This jump conditioning follows from the probability distribution, for the differential Poisson process which behaves asymptotically for small λdt as the zero-one jump law, in dt-precison,

$$\Phi_{dP(t)}(k;\lambda dt) = \mathrm{Prob}[dP(t) = k] = \left\{ \begin{array}{ll} 1 - \lambda dt, & k = 0 \\ \lambda dt, & k = 1 \\ 0, & k > 1 \end{array} \right\} + \mathrm{O}^2(\lambda dt), \quad (4.31)$$

so that $dP(t)$ behaves as an **indicator function of the jump counter k**.

The neglected error is $\mathrm{O}^2(dt) = \mathrm{o}(dt)$, i.e., $dP(t) = 0$ with asymptotic probability $(1 - \lambda dt)$ if there is no jump and $dP(t) = 1$ with asymptotic probability (λdt) if there is a jump, while multiple jumps are likely to be negligible.

Thus, the change of a composite function of the state process $X(t)$, $dF(X(t), t)$, can be decomposed into the sum of continuous and discontinuous changes.

The function $F(x, t)$ is assumed to be at least twice continuously differentiable in x and once in t.

Due to the nonsmoothness, a two-term Taylor approximation from continuous calculus yields, with subscripts denoting partial derivatives, the continuous change

$$\begin{aligned} d_{(\mathrm{cont})}F(X(t),t) \simeq\, & F_t(X(t),t)dt \\ & + F_x(X(t),t)d_{(\mathrm{cont})}X(t) \\ & + \tfrac{1}{2}F_{xx}(X(t),t)(d_{(\mathrm{cont})}X(t))^2, \end{aligned}$$

which would be the chain rule for the compound function $F(X(t),t)$ of a deterministic function $X(t)$ with the nonsmooth property.

The discontinuous change follows from the transformation of the jump in $X(t)$ at time t given in the previous section to the jump in the composite function $Y(t) = F(X(t), t)$,

$$d_{(\text{jump})}F(X(t), t) = (F(X(t) + h(X(t), t), t) - F(X(t), t))\, dP(t),$$

using the jump

$$[X](t) = h(X(t), t)dP(t)$$

and the continuity of F in t, such that when there is a jump at time T_k in $dP(t)$, the jump in F is evaluated at the prejump time T_k^-; else the discontinuous contribution is zero.

Combining the continuous and discontinuous process changes while neglecting nonzero terms of $\mathrm{o}(dt)$ in the mean square limit sense yields

$$\begin{aligned}
dF(X(t),t) &= F(X(t)+dX(t),t+dt)-F(X(t),t) \\
&\overset{\mathrm{dt}}{=} F_t(X(t),t)dt+F_x(X(t),t) \\
&\quad \cdot(f(X(t),t)dt+g(X(t),t)dW(t)) \\
&\quad +\frac{1}{2}F_{xx}(X(t),t)\cdot g^2(X(t),t)dt \\
&\quad +(F(X(t)+h(X(t),t),t) \\
&\quad -F(X(t),t))dP(t).
\end{aligned} \tag{4.32}$$

Rewriting (4.32) slightly leads to the final statement of the Itô stochastic chain rule for jump-diffusions with simple Poisson jumps.

Rule 4.2 Jump-Diffusion Chain Rule or Itô's Lemma with Jumps:

Let $F(x,t)$ be twice continuously differentiable in x and once in t.

$$\begin{aligned} dF(X(t),t) \overset{\mathrm{dt}}{=} & \left(F_t + fF_x + \tfrac{1}{2}g^2F_{xx}\right)(X(t),t)dt \\ & + (gF_x)(X(t),t)dW(t) \\ & + (F(X(t)+h(X(t),t),t) \\ & - F(X(t),t))\, dP(t). \end{aligned} \tag{4.33}$$

Here, to summarize, it is assumed that the Wiener process is **independent** of the Poisson processes and that the quadratic differential Wiener process is replaced with $(dW)^2(t) \stackrel{\mathrm{dt}}{=} dt$, its mean. Thus, the part of the $\mathrm{O}(dt)$ change in dF due to the Wiener process requires a second derivative beyond the regular calculus first derivative Taylor approximation and thus the nonsmooth Wiener property plays a strong role. The second derivative term is a diffusion term and hence the **Wiener process, $dW(t)$, or its extension to the Gaussian process, $\mu(t)dt + \sigma(t)dW(t)$, is called a diffusion process**. However, the motivations for stochastic diffusions and physical diffusions are quite different, but they both lead to diffusion equations.

The jump term uses the **zero-one jump indicator property** of $dP(t)$, so

$$\begin{aligned}[F](X(t),t) &= F(X(t)+[X](t),t) - F(X(t),t) \\ &= F(X(t)+h(X(t),t)dP(t),t) \\ &\quad -F(X(t),t) \\ &= (F(X(t)+h(X(t),t),t) \\ &\quad -F(X(t),t))\, dP(t)\end{aligned}$$

to pass the jump differential $dP(t)$ from the state argument of $F(x,t)$ to a multiplying factor of the potential jump difference $F(x+h(x,t),t) - F(x,t)$.

To express this indicator property better and simpler, replace **F(X(t),t)** by $\mathcal{F}(X(t))$ and $h(X(t),t)$ by $J(t)$, then

$$
\begin{aligned}
[\mathcal{F}](X(t)) &= \mathcal{F}(X(t)+J(t)dP(t)) - \mathcal{F}(X(t)) \\
&= \left\{ \begin{matrix} \mathcal{F}(X(t)) - \mathcal{F}(X(t)) = 0, & dP(t) = 0 \\ \mathcal{F}(X(t)+J(t)) - \mathcal{F}(X(t), & dP(t) = 1 \end{matrix} \right\}, \\
&= (\mathcal{F}(X(t)+J(t)) - \mathcal{F}(X(t))dP(t),
\end{aligned}
$$

since the last line is equivalent to the line before it by the **zero-one jump law**.

If there is a jump at $t = T_k$, then $dP(t)$ produces a change in the arguments $(X(t), t)$ of both F and h to $(x, t) = (X(T_k^-), T_k^-)$.

If F and h are continuous in the explicit t-arguments, then $(x, t) = (X(T_k^-), T_k)$ can be used.

However, in some of the more abstract books, through a a failure to understand that the Poisson process, $dP(t)$, picks the the jump-time that goes only into the jump terms $(X(t^-)\nu(t^-)dP(t)$ here) of the jump-diffusion models, but arguments like $(X(t^-), t^-)$ appear in the diffusion terms where they do not belong, since there the cumulative behavior of the SDE automatically takes the new jump information into account. Grade as D$^-$ ($\smile$).

4.2.2 Linear Jump-Diffusion SDEs

Let the linear jump **and** diffusion SDEs be combined into a single jump-diffusion SDE,

$$dX(t) = X(t)(\mu(t)dt + \sigma(t)dW(t) + \nu(t)dP(t)), \quad (4.34)$$

where $X(t_0) = x_0 > 0$ with probability one (this is for specificity, but only $x_0 \neq 0$ is sufficient to avoid a change in sign, yet for financial assets usually only $x_0 > 0$ makes sense). The set of coefficients $\{\mu(t), \sigma(t), \nu(t), \lambda(t)\}$ is assumed to be bounded and integrable, with $\nu(t) > -1$ (otherwise, positivity of $X(t)$ cannot be maintained) and $\sigma(t) > 0$ (for consistency with the interpretation of $\sigma(t)$ as a standard deviation coefficient of the diffusion process).

The logarithmic transformation of the state process $\boldsymbol{Y(t) = \ln(X(t))}$ transforms away the state from the right hand side of the SDE using the jump-diffusion chain rule (4.33) and the first two logarithmic derivatives, so

$$\begin{aligned} \boldsymbol{dY(t)} &\boldsymbol{= (\mu(t) - \sigma^2(t)/2)dt + \sigma(t)dW(t)} \\ &\boldsymbol{+ \ln(1 + \nu(t))dP(t).} \end{aligned} \tag{4.35}$$

The SDE (4.35) is a linear combination of the deterministic, diffusion and jump processes with deterministic time-dependent coefficients.

Recall that here the jump chain rule is

$$\boldsymbol{[Y](t) = \ln(X(t) + X(t)\nu(t)) - \ln(X(t)) \stackrel{\text{lol}}{=} \ln(1 + \nu(t)),}$$

since by the **law of logarithms (LOL)** $\boldsymbol{\ln(a) - \ln(b) = \ln(a/b)}$, for positive $\boldsymbol{a}$ & $\boldsymbol{b}$, so $\boldsymbol{X(t)}$ cancels. **{Do not confuse "LOL" with a texting abbreviation!}**

The SDE can be immediately, but formally integrated, to yield

$$Y(t) = y_0 + \int_{t_0}^{t} \Big((\mu(s) - \sigma^2(s)/2) ds + \sigma(s) dW(s) + \ln(1+\nu(s)) dP(s) \Big), \tag{4.36}$$

where $y_0 = \ln(x_0)$, recall that $x_0 > 0$.

Inverting logarithmic state $Y(t)$ back to the original state

$$X(t) = \exp(Y(t))$$

leads to

$$X(t) = x_0 \exp\left(\int_{t_0}^{t} \Big((\mu(s) - \sigma^2(s)/2) ds + \sigma(s) dW(s) + \ln(1+\nu(s)) dP(s) \Big) \right). \tag{4.37}$$

- **Linear Jump-Diffusion SDEs with Constant Coefficients**:

For the special case of constant rate coefficients, $\mu(t) = \mu_0, \sigma(t) = \sigma_0, \nu(t) = \nu_0$ and $\lambda(t) = \lambda_0$, also setting $t_0 = 0$, leads to the SDE

$$dX(t) = X(t)\left(\mu_0 dt + \sigma_0 dW(t) + \nu_0 dP(t)\right), \quad (4.38)$$

$X(t_0) = x_0 > 0$ with probability one with solution,

$$\begin{aligned} X(t) &= x_0 \exp\left((\mu_0 - \sigma_0^2/2)t + \sigma_0 W(t)\right. \\ &\quad \left. + \ln(1+\nu_0)P(t)\right) \quad (4.39) \\ &= x_0(1+\nu_0)^{P(t)} \exp\left((\mu_0 - \sigma_0^2/2)t + \sigma_0 W(t)\right), \end{aligned}$$

applying the logarithm-exponential inverse property.

Using the density $\phi_{W(t)}(w)$ for the diffusion $W(t)$ on L1-p23 and the discrete distribution $\Phi_{P(t)}(k) = p_k(\lambda_0 t)$ on L1-p42 for the jump process $P(t)$, together with the pairwise independence of the two processes, the state expectation can be found directly as

$$\begin{aligned}\mathrm{E}[X(t)] = & \, x_0 e^{(\mu_0-\sigma_0^2/2)t} e^{-\lambda_0 t} \sum_{k=0}^{\infty} \frac{(\lambda_0 t)^k}{k!}(1+\nu_0)^k \\ & \cdot \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} e^{-w^2/(2t)} e^{\sigma_0 w} dw \\ = & \, x_0 e^{\mu_0 t} e^{-\lambda_0 t} e^{\lambda_0 t(1+\nu_0)} = x_0 e^{(\mu_0+\lambda_0\nu_0)t},\end{aligned} \tag{4.40}$$

where the **exponential series** and **completing the square** technique have been used.

It is interesting to note that the conditional infinitesimal expectation relative to the $X(t)$ for this constant coefficient case is

$$\mathrm{E}[dX(t)|X(t)]/X(t) = (\mu_0 + \lambda_0\nu_0)dt,$$

provided that the given condition value $X(t) \neq 0$, which means that if the above infinitesimal expected result is interpreted as implying the expected rate, then the state expectation in (4.40) is the same result as for the equivalent deterministic process.

For more on this **quasi-deterministic equivalence** for linear stochastic processes, see Hanson and Ryan (1989).

Using similar applications of the same techniques, the state variance is computed to be

$$\begin{aligned}\mathrm{Var}[X(t)] &= \mathrm{E}[(X(t)-\mathrm{E}[X(t)])^2] \\ &= \mathrm{E}[X^2(t)] - \mathrm{E}^2[X(t)] \\ &= x_0^2 e^{2(\mu_0-\sigma_0^2/2)t}\Big(\mathrm{E}\left[e^{2\sigma_0 W(t)}(1+\nu_0)^{2P(t)}\right] \\ &\quad -\mathrm{E}^2\left[e^{\sigma_0 W(t)}(1+\nu_0)^{P(t)}\right]\Big) \\ &= x_0^2 e^{2(\mu_0-\sigma_0^2/2)t}\Big(e^{2\sigma_0^2 t}e^{\lambda_0 t((1+\nu_0)^2-1)} \\ &\quad -e^{\sigma_0^2 t}e^{2\lambda_0\nu_0 t}\Big) \\ &= x_0^2 e^{2(\mu_0+\lambda_0\nu_0)t}\left(e^{(\sigma_0^2+\lambda_0\nu_0^2)t}-1\right) \\ &= \mathrm{E}^2[X(t)]\left(e^{(\sigma_0^2+\lambda_0\nu_0^2)t}-1\right).\end{aligned} \tag{4.41}$$

The conditional infinitesimal variance relative to the square of the state, in this constant coefficient case, is

$$\mathrm{Var}[dX(t)|X(t)]/X^2(t) = (\sigma_0^2 + \lambda_0\nu_0^2)dt,$$

provided $X(t) \neq 0$, which in turn is the time integral of the exponent, $(\sigma_0^2 + \lambda_0\nu_0^2)t$, in the last line of (4.41) and since this exponent must be positive ($\lambda_0 > 0$), ensuring exponential amplification in time relative to the expectation exponential with exponent $((\mu_0 + \lambda_0\nu_0)t)$, which could be of any sign.

The usual measure of the relative changes of a random variable is called the **coefficient of variation**, which here is

$$\mathrm{CV}[X(t)] \equiv \frac{\sqrt{\mathrm{Var}[X(t)]}}{\mathrm{E}[X(t)]} = \sqrt{e^{(\sigma_0^2+\lambda_0\nu_0^2)t} - 1}, \quad (4.42)$$

provided $X(t) \neq 0$, which grows exponentially with time t. The $\mathrm{CV}[X(t)]$ is often used in the sciences to represent results, due to its dimensionless form. The dimensionless form makes it easier to pick out general trends or properties, especially if the $\mathrm{CV}[X(t)]$ can be distilled into something very simple.

The **probability density for the solution $X(t)$** in (4.39) in the case of the constant coefficient, linear jump-diffusion SDE can be found by application of the **law of total probability** (B.92) and the **probability inversion principle** in Lemma (B.19).

Thus, assuming $x_0 > 0$ and $\sigma_0 > 0$, with $\ln'(x) > 0$,

$$\Phi_{X(t)}(x) \equiv \text{Prob}[X(t) \le x]$$
$$= \sum_{k=0}^{\infty} \text{Prob}\Big[x_0 e^{(\mu_0 - 0.5\sigma_0^2)t + \sigma_0 W(t)}(1+\nu_0)^{P(t)}$$
$$\le x | P(t) = k] \cdot \text{Prob}[P(t) = k]$$
$$= \sum_{k=0}^{\infty} p_k(\lambda_0 t)\, \text{Prob}\Big[x_0 e^{(\mu_0 - 0.5\sigma_0^2)t + \sigma_0 W(t)}(1+\nu_0)^k]$$
$$\le x]$$
$$= \sum_{k=0}^{\infty} p_k(\lambda_0 t)\, \text{Prob}[W(t) \le (\ln(x/x_0)$$
$$-(\mu_0 - 0.5\sigma_0^2)t - k\ln(1+\nu_0)\big)/\sigma_0\big]$$
$$= \sum_{k=0}^{\infty} p_k(\lambda_0 t)\, \Phi_{W(t)}\big(\big(\ln(x/x_0) - (\mu_0 - 0.5\sigma_0^2)t$$
$$-k\ln(1+\nu_0)\big)/\sigma_0\big)$$
$$= \sum_{k=0}^{\infty} p_k(\lambda_0 t)\, \Phi_n\big(\big(\ln(x/x_0) - (\mu_0 - 0.5\sigma_0^2)t$$
$$-k\ln(1+\nu_0)\big)/\sigma_0; 0, t\big)$$
$$= \sum_{k=0}^{\infty} p_k(\lambda_0 t)\, \Phi_n\big(\ln(x); \ln(x_0) + (\mu_0 - 0.5\sigma_0^2)t$$
$$+k\ln(1+\nu_0), \sigma_0^2 t\big),$$

where $\Phi_{W(t)}$ is the distribution function of $W(t)$ in (B.22) given in terms of the normal distribution function Φ_n in (B.18) The last step again follows from the conversion identity from standard to general normal distribution, given in Exercise 9 on page B70. Thus, we have just proven the following jump-diffusion probability distribution theorem for the linear constant coefficient SDE by elementary probability principles.

Theorem 4.2.1 Jump-Diffusion Probability Distribution for Linear Constant-Coefficient SDE.

Let $X(t)$ *formally satisfy the scalar, linear, constant coefficient SDE* (4.38) *with initial condition* $X(0) = x_0 > 0$. *Then for each value of the jump counter* k, *the distribution is a sequence of distributions,*

$$\Phi_{X(t)}(x) = \sum_{k=0}^{\infty} p_k(\lambda_0 t)\Phi^{(k)}_{X(t)}(x),$$

where each term of the sequence has the form

$$\Phi^{(k)}_{X(t)}(x) = \Phi_n\left(\ln(x); \mu^{(k)}_n(t), \sigma^2_n(t)\right),$$

i.e., is a lognormal distribution (B.30) with normal mean

$$\mu^{(k)}_n(t) \equiv \ln(x_0) + \left(\mu_0 - 0.5\sigma_0^2\right)t + k\ln(1+\nu_0)$$

and normal variance

$$\sigma^2_n(t) \equiv \sigma^2_0 t.$$

For each k the logarithm of the solution $X(t)$ has a general normal distribution, where the lognormal moment formulas are given in the ***Properties B.20****. The probability density of $X(t)$ is found by the regular chain-rule differentiating the distribution to yield*

$$\begin{aligned}\phi_{X(t)}(x) = & \sum_{k=0}^{\infty} p_k(\lambda_0 t) \\ & \cdot x^{-1}\phi_n\left(\ln(x); \mu_n^{(k)}(t), \sigma_n^2(t)\right)\end{aligned} \quad (4.43)$$

for $x > 0$, such that $\phi_{X(t)}(0) \equiv \phi_{X(t)}(0^+) = 0$.

As $\exp(\ln(x)) = x \to 0^+$, $\ln(x) \to -\infty$,
$x^{-1}\phi_n \to C\exp(-0.5\ln^2(x)/\sigma_n^2(t) - \kappa(t)\ln(x) - \alpha(t))$
$\to \exp(-0.5\ln^2(x)/\sigma_n^2(t)) \to 0^+$, where
$\kappa(t) = 1 - \mu_n^{(k)}(t)/\sigma_n^2(t)$ and
$\alpha(t) = 0.5(\mu_n^{(k)})^2(t)/\sigma_n^2(t)$.

Remarks

- The fact $\phi_{X(t)}(0) \equiv \phi_{X(t)}(0^+) = 0$ is true because for the limit as $x \longrightarrow 0^+$, the order $-\ln^2(x)$ in the normal exponent dominates the $-\ln(x)$ from the algebraic pole $1/x$.
- For each k, the normal mean is shifted by an amount $\ln(1+\nu_0)$ and is weighted by the Poisson jump counting probability $p_k(\lambda_0 t) = \exp(-\lambda_0 t)(\lambda_0 t)^k/k!$, so the contributions decay like those of the exponential series due to the tremendous growth of the factorial function eventually dominating $(\lambda_0 t)^k$.