FinM 345/Stat 390 Stochastic Calculus, Autumn 2009

Floyd B. Hanson, Visiting Professor

Email: fhanson@uchicago.edu

Master of Science in Financial Mathematics Program
University of Chicago

Lecture 7 (from Chicago)

Distributions and Financial Applications

6:30-9:30 pm, 09 November 2009 at Kent 120 in Chicago

7:30-10:30 pm, 09 November 2009 at UBS Stamford

7:30-10:30 am, 10 November 2009 at Spring in Singapore

FinM 345 Stochastic Calculus:

7. Compound Jump-Diffusion Distribution and Applications in Financial Engineering:

• *7.1. Compound Jump-Diffusion Distribution:*

• 7.1.1 Distribution of Increment Log-Process:

Theorem 7.1. Distribution of the State Increment Logarithm Process for Linear Mark-Jump-Diffusion SDE:
Let the logarithm-transform jump-amplitude be $\ln(1+\nu(t,q))=q$. *Then the increment of the logarithm process* $Y(t)=\ln(X(t))$, *assuming* $X(t_0)=x_0>0$ *and the jump-count increment, approximately satisfies*

$$\Delta Y(t) \simeq \mu_{\mathrm{ld}}(t)\Delta t+\sigma_d(t)\Delta W(t)+\sum_{j=1}^{\Delta P(t;Q)} Q_j \qquad (7.1)$$

for sufficiently small Δt,

where $\mu_{\rm ld}(t) \equiv \mu_d(t) - \sigma_d^2(t)/2$ is the log-diffusion (LD) drift, $\sigma_d > 0$ and the Q_j are pairwise IID jump marks for $P(s;Q)$ for $s \in [t, t+\Delta t)$, counting only jumps associated with $\Delta P(t;Q)$ given $P(t;Q)$, with common density $\phi_Q(q)$. The Q_j are independent of both $\Delta P(t;Q)$ and $\Delta W(t)$.

Then the distribution of the log-process $Y(t)$ is the Poisson sum of nested convolutions

$$\Phi_{\Delta Y(t)}(x) \simeq \sum_{k=0}^{\infty} p_k(\lambda(t)\Delta t)\left(\Phi_{\Delta G(t)}(*\phi_Q)^k\right)(x), \quad (7.2)$$

where $\Delta G(t) \equiv \mu_{\rm ld}(t)\Delta t + \sigma_d(t)\Delta W(t)$ is the incremental Gaussian process and $(\Phi_{\Delta G(t)}(*\phi_Q)^k)(x)$ denotes a **convolution** of one distribution with k identical *densities ϕ_Q. The corresponding log-process density is*

$$\phi_{\Delta Y(t)}(x) \simeq \sum_{k=0}^{\infty} p_k(\lambda(t)\Delta t)\left(\phi_{\Delta G(t)}(*\phi_Q)^k\right)(x). \quad (7.3)$$

Proof: By the law of total probability (B.92), the distribution of the log-jump-diffusion $\Delta Y(t) \simeq \Delta G(t) + \sum_j^{\Delta P(t)} Q_j$, dropping the Q parameter in $\Delta P(t;Q)$ to simplify, is

$$\begin{aligned}
\Phi_{\Delta Y(t)}(x) &\equiv \mathrm{Prob}[\Delta Y(t) \le x] \\
&= \mathrm{Prob}\left[\Delta G(t) + \textstyle\sum_{j=1}^{\Delta P(t)} Q_j \le x\right] \\
&\overset{\mathrm{ltp}}{=} \textstyle\sum_{k=0}^{\infty} \mathrm{Prob}\Big[\Delta G(t) \\
&\qquad + \textstyle\sum_{j=1}^{\Delta P(t)} Q_j \le x \Big| \Delta P(t) = k\Big] \\
&\qquad \cdot \mathrm{Prob}[\Delta P(t) = k] \\
&= \sum_{k=0}^{\infty} p_k(\lambda(t)\Delta t)\Phi^{(k)}(x),
\end{aligned} \quad (7.4)$$

where $p_k(\lambda(t)\Delta t)$ is the Poisson distribution with parameter $\lambda(t)\Delta t$

and the k-jump distribution is

$$\Phi^{(k)}(x) \equiv \text{Prob}\left[\Delta G(t) + \sum_{j=1}^{k} Q_j \le x\right].$$

For each discrete condition $\Delta P(t) = k$, $\Delta Y(t)$ is the sum of $k+1$ terms, the normally distributed Gaussian diffusion part $\Delta G(t) = \mu_{\text{ld}}(t)\Delta t + \sigma_d(t)\Delta W(t)$ and the Poisson counting sum $\sum_{j=1}^{k} Q_j$, where the marks Q_j are assumed to be **IID** but otherwise distributed with density $\phi_Q(q)$, while independent of the diffusion and the Poisson counting differential process $\Delta P(t)$. Using the fact that $\Delta W(t)$ is normally distributed with zero-mean and Δt-variance,

$$
\begin{aligned}
\Phi_{\Delta G(t)}(x) &= \mathrm{Prob}[\Delta G(t) \le x] \\
&= \mathrm{Prob}[\mu_{\mathrm{ld}}(t)\Delta t + \sigma_d(t)\Delta W(t) \le x] \\
&= \mathrm{Prob}[\Delta W(t) \le (x - \mu_{\mathrm{ld}}(t)\Delta t)/\sigma_d(t)] \\
&= \Phi_{\Delta W(t)}((x - \mu_{\mathrm{ld}}(t)\Delta t)/\sigma_d(t)) \\
&= \Phi_n((x - \mu_{\mathrm{ld}}(t)\Delta t)/\sigma_d(t); 0, \Delta t) \\
&= \Phi_n(x; \mu_{\mathrm{ld}}(t)\Delta t, \sigma_d^2(t)\Delta t),
\end{aligned}
$$

provided $\sigma_d(t) > 0$, while also using identities for normal distributions, where $\Phi_n(x; \mu, \sigma^2)$ denotes the normal distribution with mean μ and variance σ^2.

Since $\Phi^{(k)}$ is the distribution for the sum of $k+1$ independent random variables, with one normally distributed random variable and k IID jump marks Q_j for each k, $\Phi^{(k)}$ will be the **nested convolutions** as given in (B.100), i.e.,

$$\phi_{X_1+X_2+\cdots+X_n}(z) = (\phi_{X_1} * \phi_{X_2} * \ldots * \phi_{X_n})(z)$$
$$= \left\{ \begin{matrix} ((\ldots(\phi_{X_1} * \phi_{X_2}) * \ldots * \phi_{X_{n-1}}) * \phi_{X_n})(z) \\ (\phi_{X_1} * (\phi_{X_2} * \ldots * (\phi_{X_{n-1}} * \phi_{X_n}) \ldots))(z) \end{matrix} \right\},$$

where the **convolution of a distribution or density $f(y)$** and a density $\phi(x)$ be

$$(f * \phi)(z) \equiv \int_{-\infty}^{+\infty} f(z-x)\phi(x)dx \tag{7.5}$$

provided the integral exists. The convolution arises when finding the distribution for a sum of random variables, e.g., $\Phi^{(0)} = \Phi_{\Delta G(t)}$, while $\Phi^{(1)}$ is sum of the Gaussian process and a one-jump Poisson process, $\Delta G(t) + \Delta J_1(t)$, say.

Upon expanding in convolutions starting from the distribution for the random variable $\Delta G(t)$ and the kth Poisson counting sum

$$J_k \equiv \sum_{j=1}^{k} Q_j,$$

we get

$$\Phi^{(k)}(x) = \left(\Phi_{\Delta G(t)} * \phi_{J_k}\right)(x) = \left(\Phi_{\Delta G(t)} \prod_{i=1}^{k} (*\phi_{Q_i})\right)(x)$$
$$= \left(\Phi_{\Delta G(t)} (*\phi_Q)^k\right)(x),$$

using the identically distributed property of the Q_i's and the compact convolution operator notation

$$\left(\Phi_{\Delta G(t)} \prod_{i=1}^{k} (*\phi_{Q_i})\right)(x) = ((\cdots(\Phi_{\Delta G(t)} * \phi_{Q_1}) * \cdots * \phi_{Q_k})(x),$$

which collapses to the operator power form for IID marks analogously to reduction of a product to a power, $\prod_{i=1}^{k} c = c^k$, for some constant c.

Substituting the distribution into the law of total probability form (7.4), the desired result is (7.2), which when differentiated with respect to x yields the kth density $\phi_{\Delta Y(t)}(x)$ in (7.3). $\square$

Remark 7.1: *Several specialized variations of this theorem are found in Hanson and Westman [2002a,2002b], to these papers are made here.*

Corollary 7.1. Density of Linear Jump-Diffusion with Log-Normally Distributed Jump-Amplitudes (not recommended due to thin, not fat tails):

Let $\boldsymbol{X}(\boldsymbol{t})$ *be a linear jump-diffusion satisfying SDE* (7.1) *or ((5.69), p. 153, textbook) and let the jump-amplitude mark* $\boldsymbol{Q}$ *be normally distributed such that*

$$\phi_Q(x;t) = \phi_n(x;\mu_j(t),\sigma_j^2(t)) \tag{7.6}$$

with jump (j) mean $\mu_j(t) = \mathrm{E}[Q]$ *and jump (j) variance* $\sigma_j^2(t) = \mathrm{Var}[Q]$*. Then the jump-diffusion density of the log-process* $\boldsymbol{Y}(\boldsymbol{t})$ *is*

$$\begin{aligned}\phi_{\Delta Y(t)}(x) = \sum_{k=0}^{\infty} p_k(\lambda(t)\Delta t)\phi_n(x;\mu_{\mathrm{ld}}(t)\Delta t + k\mu_j(t), \\ \sigma_d^2(t)\Delta t + k\sigma_j^2(t)).\end{aligned} \tag{7.7}$$

Proof: By (B.101) the convolution of two normal densities is a normal distribution with a mean that is the sum of the means and a variance that is the sum of the variances. Similarly, by the induction exercise result in (B.196), the pairwise convolution of one normally distributed diffusion process $\Delta G(t) = \mu_{\rm ld}(t)\Delta t + \sigma_d(t)\Delta W(t)$ density and k random mark Q_i densities ϕ_Q for $i = 1\!:\!k$ will be a normal density whose mean is the sum of the $k+1$ means and whose variance is the sum of the $k+1$ variances.

Thus, starting with the result (7.4) and then applying (B.196),

$$\begin{aligned}
\phi_{\Delta Y(t)}(x) &= \sum_{k=0}^{\infty} p_k(\lambda(t)\Delta t)\left(\phi_{\Delta G(t)}\,(*\phi_Q)^k\right)(x) \\
&= \sum_{k=0}^{\infty} p_k(\lambda(t)\Delta t)\phi_n\Big(x;\mu_{\mathrm{ld}}(t)\Delta t+\sum_{i=1}^{k}\mu_j(t), \\
&\qquad\qquad \sigma_d^2(t)\Delta t+\sum_{i=1}^{k}\sigma_j^2(t)\Big) \\
&= \sum_{k=0}^{\infty} p_k(\lambda(t)\Delta t)\phi_n(x;\mu_{\mathrm{ld}}(t)\Delta t+k\mu_j(t), \\
&\qquad\qquad \sigma_d^2(t)\Delta t+k\sigma_j^2(t)). \qquad \square
\end{aligned}$$

Remarks 7.2: ***So the density of the log-process is a mixture or sum of normal densities with shift means and variances.***

The normal jump-amplitude jump-diffusion distribution has been used in financial applications, initially by Merton (1976) and then by others such as Torben Andersen, Benzoni and Lund (2002) of Northwestern University and also by Hanson and Westman (2002).

Corollary 7.2. Density of Linear Jump-Diffusion with Log-Uniformly Distributed Jump-Amplitudes:
Let $X(t)$ *be a linear jump-diffusion satisfying SDE* (7.1), *and let the jump-amplitude mark* Q *be uniformly distributed as in ((5.28), L5-p58 or p. 138 textbook), i.e.,*

$$\phi_Q(q) = \frac{1}{b-a} U(q; a, b),$$

where $U(q; a, b) = I_{\{q \in [a,b]\}}$ *is the unit step function or indicator function on* $[a, b]$ *with* $a < b$. *The jump-mean is* $\mu_j(t) = (b+a)/2$ *and jump-variance is* $\sigma_j^2(t) = (b-a)^2/12$. *Then the jump-diffusion density of the increment log-process* $\Delta Y(t)$ *satisfies the general convolution form* (7.3), *i.e.,*

$$\begin{aligned} \phi_{\Delta Y(t)}(x) &= \sum_{k=1}^{\infty} p_k(\lambda(t)\Delta t) \left(\phi_{\Delta G(t)} (*\phi_Q)^k\right)(x) \\ &= \sum_{k=1}^{\infty} p_k(\lambda(t)\Delta t) \phi_{\mathrm{ujd}}^{(k)}(x), \end{aligned} \tag{7.8}$$

where $p_k(\lambda(t)\Delta t)$ *is the Poisson distribution with parameter* $\lambda(t)$.

The $\Delta G(t) = \mu_{\rm ld}(t)\Delta t + \sigma_d(t)\Delta W(t)$ *is the diffusion term and* Q *is the uniformly distributed jump-amplitude mark. The first few coefficients of* $p_k(\lambda(t)\Delta t)$ *for the uniform jump-distribution* ***(UJD)****, starting with a pure diffusion density, are*

$$\phi^{(0)}_{\rm ujd}(x) \equiv \phi_{\Delta G(t)}(x) = \phi_n(x;\mu_{\rm ld}(t)\Delta t, \sigma^2_d(t)\Delta t), \quad (7.9)$$

where $\phi_n(x;\mu_{\rm ld}(t)\Delta t, \sigma^2_d(t)\Delta t)$ *denotes the normal density with mean* $\mu_{\rm ld}(t)\Delta t$ *and variance* $\sigma_d(t)\Delta t$,

$$\begin{aligned}\phi^{(1)}_{\rm ujd}(x) &= \left(\phi_{\Delta G(t)} * \phi_Q\right)(x) \\ &= \phi_{sn}(x-b, x-a;\mu_{\rm ld}(t)\Delta t, \sigma^2_d(t)\Delta t),\end{aligned} \quad (7.10)$$

where ϕ_{sn} *is the* ***secant-normal density***

$$\begin{aligned}\phi_{sn}(x_1, x_2;\mu, \sigma^2) &\equiv \Phi_n(x_1, x_2;\mu, \sigma^2)/(x_2 - x_1) \\ &\equiv (\Phi_n(x_2;\mu, \sigma^2) - \Phi_n(x_1;\mu, \sigma^2))/(x_2 - x_1)\end{aligned} \quad (7.11)$$

with normal distribution $\Phi_n(x_1, x_2;\mu, \sigma^2)$.

Also,

$$
\begin{aligned}
\phi_{\mathrm{ujd}}^{(2)}(x) &= \left(\phi_{\Delta G(t)}(*\phi_Q)^2\right)(x) \\
&= \tfrac{2b-x+\mu_{\mathrm{ld}}(t)\Delta t}{b-a}\phi_{sn}(x-2b, x-a-b; \mu_{\mathrm{ld}}(t)\Delta t, \sigma^2(t)\Delta t) \\
&\quad + \tfrac{x-2a-\mu_{\mathrm{ld}}(t)\Delta t}{b-a}\phi_{sn}(x-a-b, x-2a; \mu_{\mathrm{ld}}(t)\Delta t, \sigma_d^2(t)\Delta t) \\
&\quad + \tfrac{\sigma_d^2(t)\Delta t}{(b-a)^2}\big(\phi_n(x-2b; \mu_{\mathrm{ld}}(t)\Delta t, \sigma_d^2(t)\Delta t) \\
&\quad -2\phi_n(x-a-b; \mu_{\mathrm{ld}}(t)\Delta t, \sigma_d^2(t)\Delta t) \\
&\quad +\phi_n(x-2a; \mu_{\mathrm{ld}}(t)\Delta t, \sigma_d^2(t)\Delta t)\big)\,. \qquad (7.21)
\end{aligned}
$$

Proof: First the finite range of the jump-amplitude uniform density is used to truncate the convolution integrals for each k using existing results for the mark convolutions, such as $\phi^{(2)}_{(\mathrm{uq})}(x) = (\phi_Q * \phi_Q)(x) = \phi_{Q_1+Q_2}(x)$ for IID marks when $k = 2$.

The case for $k = 0$ is trivial since it is given in the theorem equations (7.9).

For a $k = 1$ jump,

$$\begin{aligned}
\phi^{(1)}_{\text{ujd}}(x) \equiv (\phi_{\Delta G(t)} * \phi_Q)(x) \equiv \int_{-\infty}^{+\infty} \phi_{\Delta G(t)}(x-y)\phi_Q(y)dy \\
&= \frac{1}{b-a}\int_a^b \phi_n(x-y;\mu_{\text{ld}}(t)\Delta t, \sigma_d^2(t)\Delta t)dy \\
&= \frac{1}{b-a}\int_{x-b}^{x-a} \phi_n(z;\mu_{\text{ld}}(t)\Delta t, \sigma_d^2(t)\Delta t)dz \\
&= \frac{1}{b-a}\Phi_n(x-b, x-a;\mu_{\text{ld}}(t)\Delta t, \sigma_d^2(t)\Delta t) \\
&= \phi_{sn}(x-b, x-a;\mu_{\text{ld}}(t)\Delta t, \sigma_d^2(t)\Delta t),
\end{aligned}$$

where $-\infty < x < +\infty$, upon change of variables and use of normal density identities.

For $k = 2$ jumps, the convolution of two copies of the uniform distribution on $[a, b]$ results in a **triangular distribution** on $[2a, 2b]$ which, from exercise result (B.197), is

$$\phi_{(\mathrm{uq})}^{(2)}(x) = (\phi_Q * \phi_Q)(x) = \frac{1}{(b-a)^2} \left\{ \begin{array}{ll} x-2a, & 2a \le x < a+b \\ 2b-x, & a+b \le x \le 2b \\ 0, & otherwise \end{array} \right\}. \tag{7.12}$$

Hence,

$$
\begin{aligned}
\phi_{\text{ujd}}^{(2)}(x) &= (\phi_{\Delta G(t)} * (\phi_Q * \phi_Q))(x) \\
&= \int_{-\infty}^{+\infty} \phi_{\Delta G(t)}(x-y)(\phi_Q * \phi_Q)(y)dy \\
&= \frac{1}{(b-a)^2}\bigg(\int_{2a}^{a+b} (y-2a)\phi_{\Delta G(t)}(x-y)dy \\
&= + \int_{a+b}^{2b} (2b-y)\phi_{\Delta G(t)}(x-y)dy \bigg) \\
&= \frac{1}{(b-a)^2}\bigg(\int_{x-a-b}^{x-2a} (x-z-2a)\phi_{\Delta G(t)}(z)dz \\
&\quad + \int_{x-2b}^{x-a-b} (2b-x+z)\phi_{\Delta G(t)}(z)dz \bigg) \\
&= \frac{2b-x+\mu_{\text{ld}}(t)\Delta t}{b-a}\phi_{sn}(x-2b, x-a-b; \mu_{\text{ld}}(t)\Delta t, \sigma_d^2(t)\Delta t) \\
&\quad + \frac{x-2a-\mu_{\text{ld}}(t)\Delta t}{b-a}\phi_{sn}(x-a-b, x-2a; \mu_{\text{ld}}(t)\Delta t, \sigma_d^2(t)\Delta t) \\
&\quad + \frac{\sigma_d^2(t)\Delta t}{(b-a)^2}\big(\phi_n(x-2b; \mu_{\text{ld}}(t)\Delta t, \sigma_d^2(t)\Delta t) \\
&\quad -2\phi_n(x-a-b; \mu_{\text{ld}}(t)\Delta t, \sigma_d^2(t)\Delta t) \\
&\quad +\phi_n(x-2a; \mu_{\text{ld}}(t)\Delta t, \sigma_d^2(t)\Delta t)\big)\,,
\end{aligned}
$$

where the exact integral for the normal density has been used.

□

Remarks 7.3:

- *This density form* ϕ_{sn} *in* (7.11) *is called a* ***secant-normal density*** *since the numerator is an increment of the normal distribution and the denominator is the corresponding increment in its state arguments, i.e., a* ***secant approximation****, which here has the form* $\Delta\Phi_n/\Delta\mathbf{x}$.
- *The uniform jump-amplitude jump-diffusion distribution has been used in financial applications, initially by Hanson and Westman in (2002fmt) as a simple, but appropriate, representation of a jump component of market distributions, and some errors have been corrected have been corrected in the textbook.*

• *7.2. Applications in Financial Engineering and Mathematics:*

• 7.1.1 Some Basic Background for Options

- ***Discrete Compound Interest*** *for* m *discrete periods per year and for* n *total periods starting with a* ***present (also principal) value*** *of* $\mathbf{PV}_0$ *yields the* ***future value*** *after* i *periods at constant* ***spot interest rate*** r_0 *per year,*

$$\mathbf{FV}_i = \mathbf{PV}_0(1 + r_0/m)^i,$$

for $i = 0\!:\!n$ *periods. The inverse under the same rate is*

$$\mathbf{PV}_0 = \mathbf{FV}_i/(1 + r_0/m)^i,$$

but if a discounted loan, such as a bank gets from the Federal Reserve bank, with the amount $\mathbf{FV}_n$ *due after* n *periods, then the calculated* ***discount rate*** β_0 *is slightly different than the* ***spot interest rate*** r_0 *and the amount that the borrower receives at* $i = 0$ *is*

$$\widehat{\mathbf{PV}}_0 = \mathbf{FV}_n/(1 + \beta_0/m)^n.$$

- ***Continuous Compound Interest*** *follows from the discrete case letting* $t{=}n/m$ *be time in years and* $m \longrightarrow \infty$ *for fixed* t, *yielding the limit*[a]

$$\mathbf{FV}(t) = \mathbf{PV}(0)e^{r_0 t}.$$

Letting $B(t)=FV(t)$, *whether a bank saving account or money market fund or bond asset (technically a zero coupon bond to avoid including income here), then*

$$dB(t) = r_0\mathbf{PV}(0)e^{r_0 t}dt = r_0 B(t)dt.$$

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[a] This follows from with n=mt
$(1+r/m)^{mt} \stackrel{\text{loe}}{=} \exp(mt\ln(1+r/m)) \longrightarrow \exp(mt{\cdot}r/m + \mathrm{O}(1/m)) \longrightarrow \exp(rt)$
as $m \longrightarrow +\infty$.

- ***Opportunity costs*** *concern the benefits of considering alternate investments, so if considering a bond at initial price* $B(0)$ *at time* t *with interest rate* r_0 *in comparison with a stock at* $S(0) = B(0)$ *say for a fair comparison, with growth rate* μ_0*, then in absence of stochastic effects,*

$$dB(t) = r_0 B(t)dt \ \ \& \ \ dS(t) = \mu_0 S(t)dt,$$

with $S(t) = S(0)\exp(\mu_0 t)$*, so the relative difference is*

$$S(t)/B(t) = S(0)\exp(\mu_0 t)/(B(0)\exp(r_0 t)) \stackrel{\text{loe}}{=} \exp((\mu_0 - r_0)t).$$

Hence, the stock would be chosen if $\mu_0 > r_0$ *else the bond would be chosen. We choose the investment with the highest return,* $\Delta B(t)/B(t)$ *or* $\Delta S(t)/S(t)$ *. Such a reliable reference investment, like a riskless bond, is called a* ***numeraire*** *for the more risky stock. In the presence of risk due to stochastic effects, an investment portfolio would be weighted with less and more risky assets* ***depending on the investor risk aversion****.*

- ***Quick Options Glossary:*** *(see Hull's "trader's bible" for more.)*

  *** Financial Options:** *Investment contracts (a financial derivative, i.e., "derived" from another investment) for limiting risk of financial loss for underlying asset (e.g., common stocks; there are too many to list).*

  *** Holder:** *Buyer of stock options.*

  *** Writer:** *Seller of stock options writer of the contract.*

  *** Exercise or Strike Price:** *The contract price (K {marks a strike in bowling} or E {can be confused with expectation}) for buying or selling the underlying asset, often in discrete increments.*

  *** Exercise or Strike Time:** *The contract expiration time (T) for buying or selling the underlying asset, must be before the end of trading day, possibly restrictions on days.*

  *** Option Premium:** *The price (P_o) of the option the holder pays to the writer of the contract at $t = 0$, usual a clearing house is involved, the clearing now undergoing a lot of changes.*

* ***Call Option (Simple or Vanilla Version):*** *Option contract for the holder to buy from the writer, on or before T, an amount of the asset at price K.*
* ***Put Option (Simple or Vanilla Version):*** *Option contract for the holder to sell to the writer, on or before T, an amount of the asset at price K.*
* ***Option Payoff:*** $\mathrm{Payoff} = \max(\theta \cdot (S(t) - K), 0)$ *where for vanilla options,* $\theta = 1$ *for calls and* $\theta = -1$ *for puts; not counting the option price paid.*
* ***European Options:*** *Option contract that can only be exercised at strike time T (easiest to price, not less flexible).*
* ***American Options:*** *Option contract that can be exercised at or before strike time T (harder to price, but more flexible and common).*
* ***Options Trading:*** *For example, CBOE (Chicago Board of Options Exchange) or ISE (International Securities Exchange).*

* ***Long Options:*** *Buy and hold option.*

* ***Short Options:*** *Sell an options contract.*

* ***ATM:*** *At The Money:* $\boldsymbol{S(t) = K}$ & $\mathbf{Payoff = 0}$.

* ***ITM:*** *In The Money: Profit or* $\mathbf{Payoff > 0}$.

* ***OTM:*** *Out of The Money: Loss, technically, or* $\boldsymbol{\theta \cdot (S(t) - K)}$*, so rational holder would would walk away and we assume* $\mathbf{Payoff = 0}$*, unlike nonoptional future derivatives.*

* ***Net Profit (or Loss):*** $\mathbf{Payoff} - \boldsymbol{P_o}$ *for the holder, while its negative for the writer. (However, this is the least complicated scenario.)*

* ***Break Even Point (BEP):*** *Underlying asset price for which the net profit is zero, i.e., at* $\boldsymbol{S(t) = \mathrm{BEP} \equiv K + \theta \cdot P_o}$.

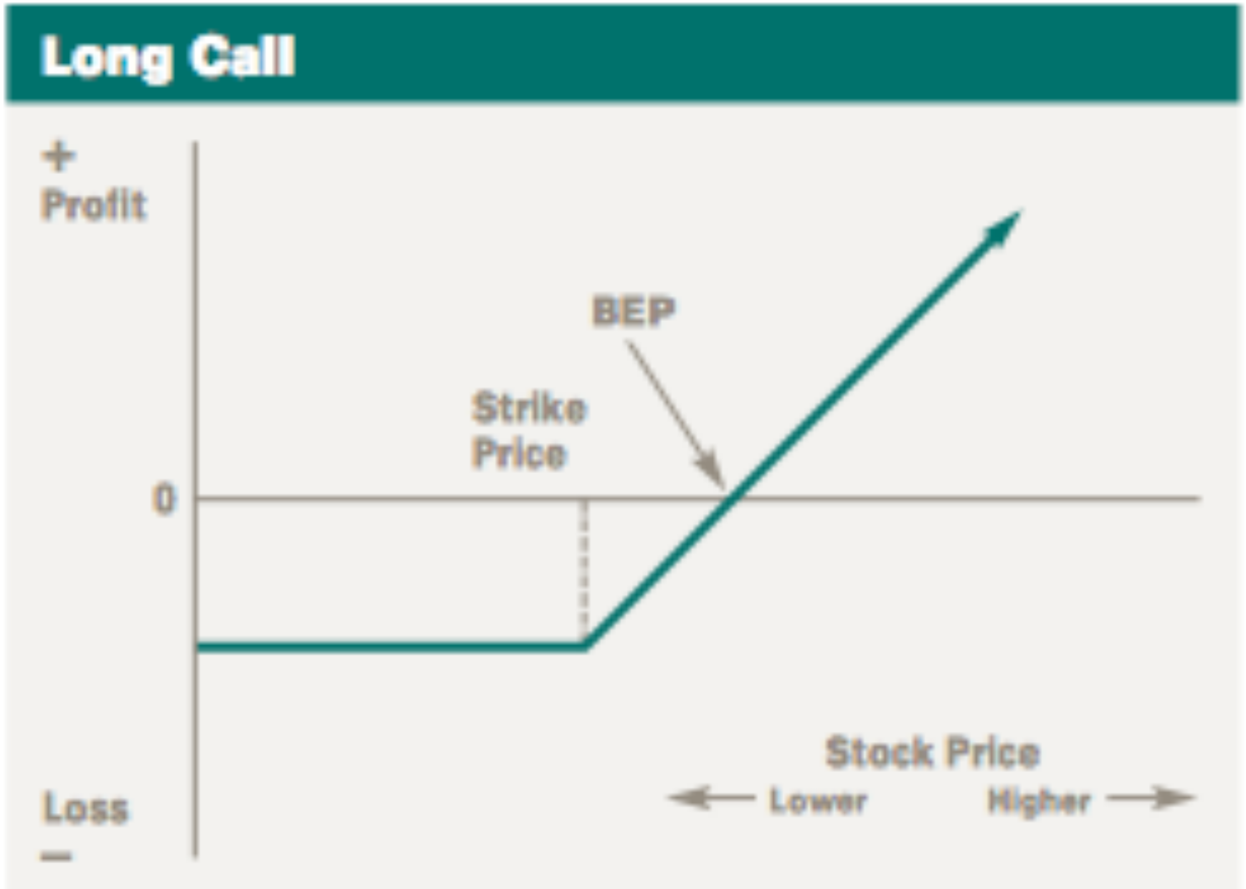

Figure 7.1: Long Call Net Profit: Bullish; Holder Bets on Gains, so Buy and Hold Call until Exercise to Buy Stock from Writer at K if $S > \mathrm{BEP} = K + P_o$, else Walk (See ***The Equity Options Stategy Guide***, Options Clearing Corporation (OCC), April 2003, p. 10).

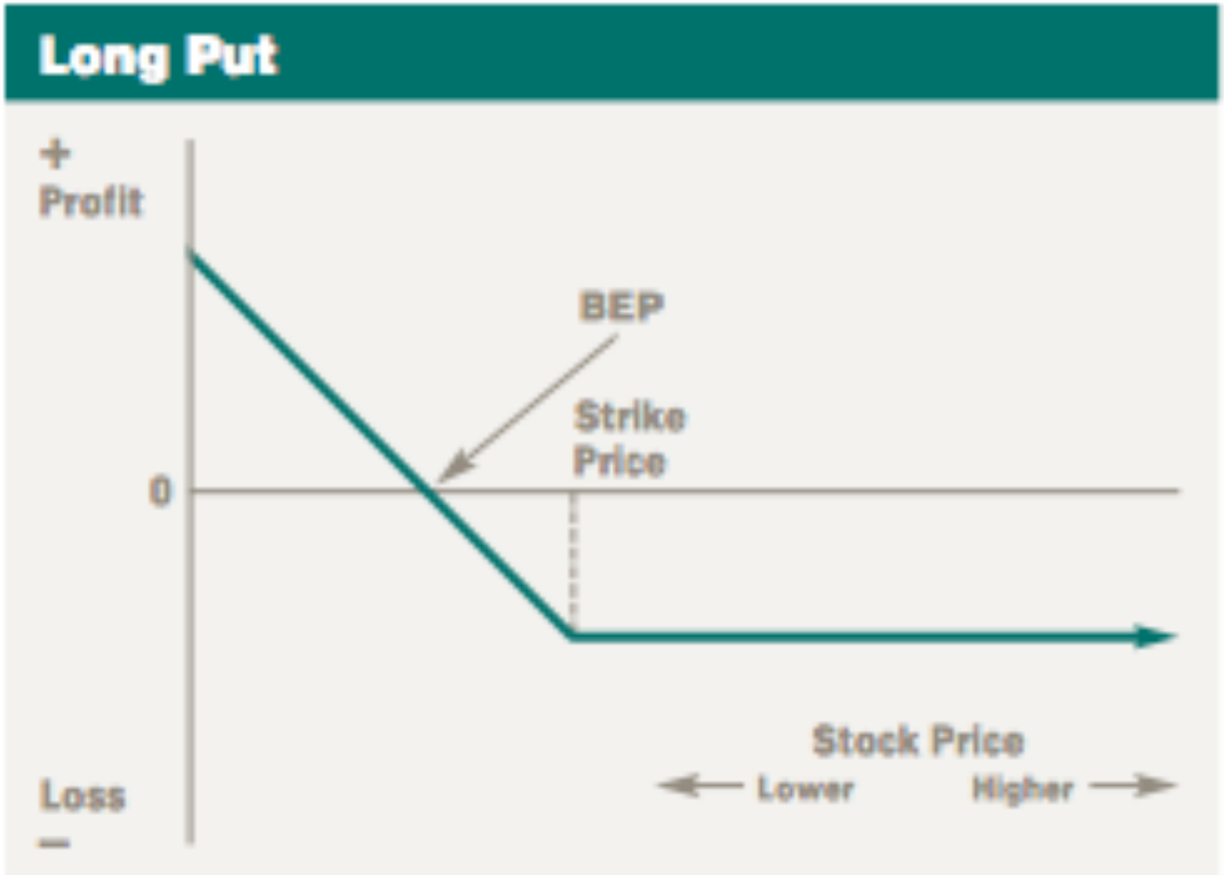

Figure 7.2: Long Put Net Profit: Bearish; Holder Bets on Loss, so Buy and Hold Put until Exercise to Sell Stock to Writer at K if $S < \text{BEP} = K - P_o$, else Walk (See *The Equity Options Stategy Guide*, Options Clearing Corporation (OCC), April 2003, p. 12).

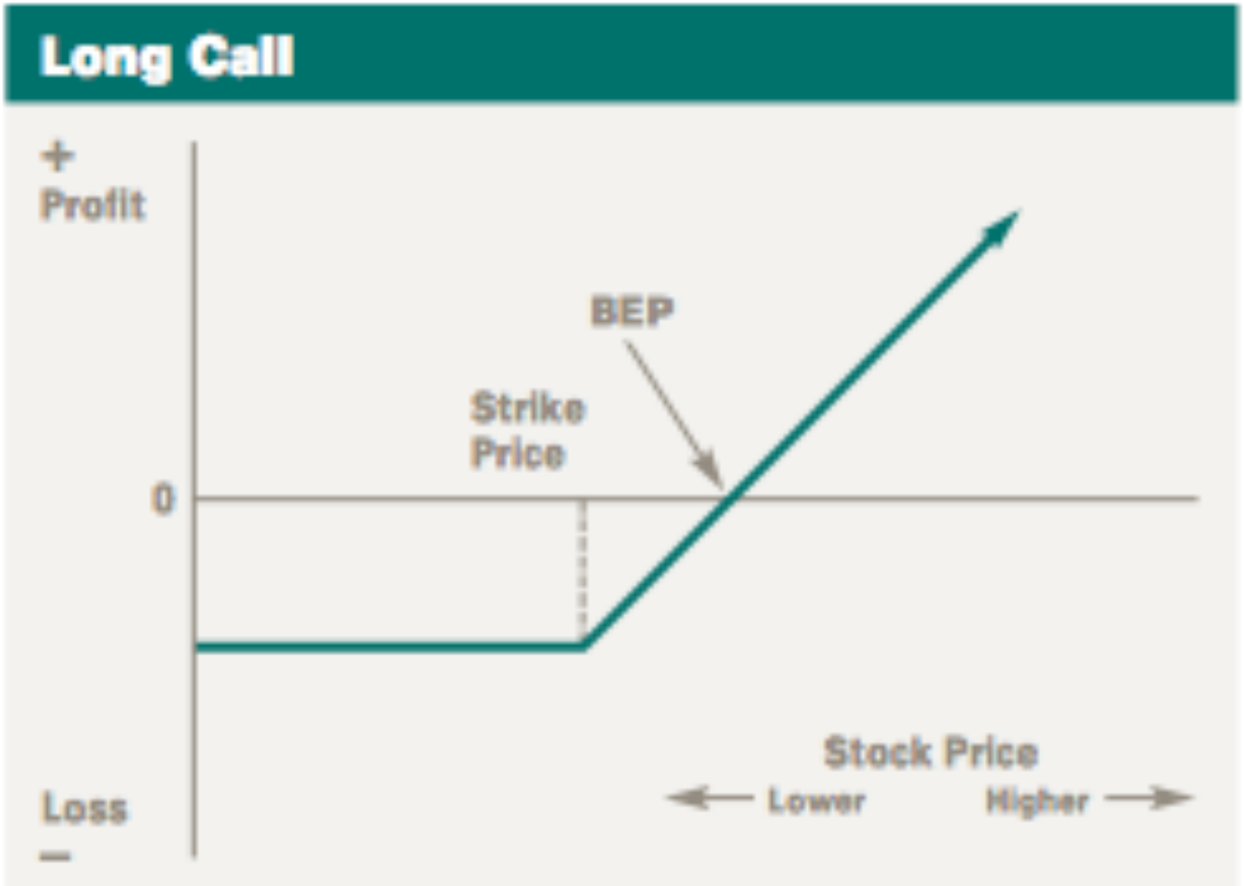

Figure 7.3: Hedged Long Put Net Profit (Same as Fig. 7.1, different reasons): Bullish; Holder Buys Put and also Hedges by Buying same Stock at $\boldsymbol{S(0)}$, so Loss is limited by $\boldsymbol{K – S(0) – P_o}$, but for $\boldsymbol{S(t) > K}$ hold on net profit $\boldsymbol{S(t) – \mathrm{BEP}}$ where $\mathbf{BEP} = \boldsymbol{S(0) + P_o}$ (See also J.C. Hull, ***Options, Futures & Other Derivatives***, 4th Edn. (not in 6th), p. 186; was IRS special tax case).

- **7.1.2 Black-Scholes Simple Option Pricing with Delta Hedging:**

The famous option formula from eliminating instantaneous volatility risk. Here, some Merton observations are used.

1. **Geometric Brownian motion SDE** for underlying asset with price $S(t)$ at t with constant coefficients $\{\mu_0, \sigma_0\}$:
$$dS(t) = S(t)(\mu_0 dt + \sigma_0 dW(t)), \quad S(0) = S_0. \tag{7.13}$$
2. **Small time increment**, $\Delta t \ll 1$, with $S\Delta E$,
$$\Delta S(t) \simeq S(t)(\mu_0 \Delta t + \sigma_0 \Delta W(t)), \quad S(0) = S_0.$$
3. **Option price** depends on underlying price and time, $Y(t) = F(S(t), t)$:
$$\Delta Y(t) = \Delta F(S(t), t)$$
$$\simeq (F_t + \mu_0 s F_s + 0.5\sigma_0^2 s^2 F_{ss})\Delta t + \sigma_0 s F_s \Delta W(t),$$
where $s = S(t)$ for brevity and all F partials are evaluated at $(S(t), t)$ and volatility risk is now $\sigma_0 F_s \Delta W(t)$. Note that all s-terms are in scale invariant.

4. **Stock and Options Portfolio** with N_s stock shares and N_f option shares with portfolio value,

$$V(t) = N_f F(S(t), t) + N_s S(t),$$

taking the riskless bond, with $\Delta B(t) \simeq r_0 B(t) \Delta t$, as optional.

5. The notorious **Self-Financing Strategy**: By the product rule, neglecting only second order changes,

$$\Delta V(t) \simeq N_f \Delta F + N_s \Delta S + F \Delta N_f + S \Delta N_s,$$

but we assume that the changes in the shares are much smaller than the changes in the prices, $F \Delta N_f + S \Delta N_s \ll N_f \Delta F + N_s \Delta S$ then the self-financing strategy is $\Delta V(t) \simeq N_f \Delta F + N_s \Delta S$ or

$$\Delta V(t) \simeq N_f((F_t + \mu_0 s F_s + 0.5\sigma_0^2 s^2 F_{ss})\Delta t$$
$$+ \sigma_0 s F_s \Delta W(t)) + N_s(\mu_0 \Delta t + \sigma_0 \Delta W(t)),$$

noting that many authors like Hull overlook this assumption.

{*Also, recall the difficulty Black and Scholes had in getting their 1973 paper published and that Merton had to hold up his 1973 companion and justification paper until B&S's paper was accepted.*}

6. **No Friction** assumption: No transaction fees and no dividends or other income to the portfolio (jumps!).
7. **Portfolio Deviation and Volatility Risk:**

$$\mathrm{Dev}[\Delta V(t)|Y,S] = \Delta V(t) - \mathrm{E}[\Delta V(t)|Y(t),S(t)=s]$$
$$= \sigma_0 s(N_f \cdot F_s + N_s)\Delta W(t).$$

8. **Elimination of and Optimal Hedge against Volatility Risk:** Select portfolio share numbers so that

$$N_f \cdot F_s + N_s \stackrel{*}{=} 0 \implies N_s^* = -N_f^* \cdot F_s \text{ or } N_s^*/N_f^* = -F_s.$$

9. $\Delta_F \equiv \partial F/\partial s = F_s$ is the **(Greek) Delta of the Portfolio** or the sensitivity of the option to the underlying stock or asset at any time t and hence for the term **Delta Hedging**.

10. If the **Total Number of Shares** $N = N_s^* + N_f^*$ then the **instantaneous Share Fractions**, provided $F_s \neq 1$, are

$$\frac{N_s^*}{N} = \frac{-F_s}{1 - F_s} \;\&\; \frac{N_f^*}{N} = \frac{1}{1 - F_s}.$$

11. Optimal Portfolio Change and Value: By eliminating N_s^*, which could be thought of as a simple control variable,

$$\Delta V^*(t) = N_f^*(\Delta F - sF_s) = N_f^*(F_t + 0.5\sigma_0^2 s^2 F_{ss})\Delta t$$

and

$$V^*(t) = N_f^*(F - sF_s).$$

Note that the volatility risk $\sigma_0 dW(t)$ and the mean rate μ_0 have been eliminated together.

12. **Arbitrage Avoidance**, i.e., in theory, a price differential profit opportunities between securities cannot last long before being discovered by other investors (cf. market equilibrium theory), so it is assumed that the market return is at the risk-free rate of r_0, thus

$$\Delta V^*(t) = r_0 V^*(t) \Delta t, \qquad (7.14)$$

so substituting for $\Delta V^*(t)$ & $V^*(t)$, also canceling out the common factor of N_f^*, leads to the desired equation for $F(s,t)$ conditioned by $S(t) = s$.

13. **Black-Scholes(-Merton) PDE of Option Pricing:**

$$F_t(s,t) + 0.5\sigma_0^2 s^2 F_{ss}(s,t) = r_0(F - sF_s)(s,t). \qquad (7.15)$$

Note that the SDE is an equation for a trajectory of the asset $S(t)$, but the PDE is for $F(s,t)$ with s and t as functionally independent variables, giving a 2-dimensional view of F over a space-time values (s,t).

14. BSM PDE Final, Exercise Conditions at $t = T$:

European Call Option:

$$F(S(T), T) = \mathcal{C}(S(T), T) = \max[S(T) - K, 0];$$

European Put Option:

$$F(S(T), T) = \widehat{\mathcal{P}}(S(T), T) = \max[K - S(T), 0];$$

$\Longrightarrow$ **Backward or Final Value PDE Problem for the BSM PDE of Option Pricing**, while the solution is the option price or premium, which is the initial value $\mathcal{C}(S_0, 0)$ or $\widehat{\mathcal{P}}(S_0, 0)$, respectively.

American Option Problems: Much more complicated, because the final value problem is a ***Moving Boundary Problem***, $F(S(\tau^*), \tau^*) = \max[\theta \cdot (S(\tau^*) - K), 0]$, where the unknown early exercise time $\tau^* \leq T$ can be determined by the smooth contact point to the payoff curve.

15. Black & Scholes Formal Solution: In the 1973 paper they gave for the European Call Option Price,

$$\mathcal{C}(s,t) = C(s,t;T,K,r_0,\sigma_0) = s\Phi_n(d_1(s,T-t,K,r_0,\sigma_0),0,1) - Ke^{-r_0(T-t)}\Phi_n(d_2(s,T-t,K,r_0,\sigma_0),0,1), \tag{7.16}$$

where the BS argument functions are

$$d_1(s,T-t,K,r_0,\sigma_0),0,1) \equiv \frac{\ln(s/K)+(r_0+\sigma_0^2/2)(T-t)}{\sigma_0\sqrt{T-t}},$$
$$d_2(s,T-t,K,r_0,\sigma_0),0,1) \equiv d_1(s,T-t,K,r_0,\sigma_0),0,1) - \sigma_0\sqrt{T-t}, \tag{7.17}$$

where $t<T$. An important feature to note is that the BS call option price has a single time dependence that is the exercise **time-to-go** $(T-t)$, due to stationarity, so the formula is good for any exercise horizon that is positive.

Also, note the payoff linear dependence on s and K is preserved by transformation to the solution, but with two different nonlinear coefficients.

The proof of that (7.16)-(7.17) is a solution of the PDE problem (7.15) plus final condition is left as an exercise, noting that this can be done by substitution, without any knowledge of how to solve the PDE.

16. **Put-Call Parity for European Options:** The relationship beween the European call and put prices depends essentially on the property of the maximum function.

Let $V_c(t)$ be a call portfolio with one call option on a share of stock plus cash in a bond at rate r_0 such that it will be worth K at $t=T$.

Let $V_{\widehat{p}}(t)$ be a put portfolio with one put option on the stock plus one share of the stock.

$\{$*Really a hedged bullish spread.*$\}$

Present value of Bond at $t = 0$ is $B_0 = Ke^{-r_0 T}$.

Future value of Bond at $t = T$ is $B(T) = K$.

Future value at T: $V_c(T) = \max[S(T) - K, 0] + B(T)$.

Future value at T: $V_{\widehat{p}}(T) = \max[K - S(T), 0] + S(T)$.

Thus, $V_{\widehat{p}}(T) = V_c(T)\ \forall\ S(T) \geq 0$, with corresponding bond value $B(T) \geq 0$, since

$$V_{\widehat{p}}(T) = \left\{\begin{matrix} K, & K \geq S(T) \\ S(T), & S(T) \geq K \end{matrix}\right\} = \max[S(T), K] = V_c(T).$$

Hence, also true at any pair $\{S(t) \geq 0, B(t) \geq 0\}$, there is **Put-Call Parity**:

$$\mathcal{C}(S(t), t) + B(t) = \widehat{\mathcal{P}}(S(t), t) + S(t) \qquad (7.18)$$

or

Put-Call Parity in terms of Premia $\{\mathcal{C}(S_0, 0), \widehat{\mathcal{P}}(S_0, 0)\}$:

$$\widehat{\mathcal{P}}(S_0, 0) = \mathcal{C}(S_0, 0) + Ke^{-r_0 T} - S_0.$$

{*Comment: For European options, it is only necessary, to compute only one of the put-call pair by the Black-Scholes formula (7.16), since the other can be computed more easily by put-call parity (7.18).*}

17. Risk-Neutral Formulation: Since the BS problem is now considered a "*toy*" model, with only diffusive noise and constant coefficients, while Merton (1973) showed that variable coefficients were not a big deal and in (1976) showed that jump noise killed the delta hedging, it is helpful to look for the qualitative features of the BS PDE formulation that could be transferable to more complicated asset dynamics, such as jump-diffusions and stochastic volatility, the so-called incomplete markets.

Reforming the BS-PDE (7.15),

$$F_t(s,t)+r_0 sF_s(s,t)+0.5\sigma_0^2 s^2 F_{ss}(s,t)=r_0 F(s,t),$$

we see that the mean rate μ_0 has been replaced by the risk-free rate r_0 on the LHS, while the term $r_0 F(s,t)$ represents that part of the no-arbitrage condition that did not come directly from the driving BS-SDE (7.13).

Note that if $\widehat{F}_t(s,t) = r_0 \widehat{F}(s,t)$, really an ODE, so $(e^{-r_0 t}\widehat{F})_t = 0 \ \Rightarrow \ \widehat{F}(s,t) = C(s)e^{r_0 t}$, and this suggest elimination of the non-derivative term by letting $G(s,t) = e^{-r_0 t}F(s,t)$, yielding the **risk-neutral form of BS-PDE**,

$$G_t(s,t) + r_0 s G_s(s,t) + 0.5\sigma_0^2 s^2 G_{ss}(s,t) = 0, \quad (7.19)$$

corresponding to a hypothetical risk-neutral (RN) SDE,

$$dS^{(\mathrm{rn})}(t) = S^{(\mathrm{rn})}(t)(r_0 dt + \sigma_0 dW(t)). \quad (7.20)$$

The corresponding European final condition for PDE (7.19) is

$$G(S(T),T) = e^{-r_0 T}\max[\theta \cdot (S(T) - K), 0],$$

a payoff discounted at rate r_0 back to zero. The problem with this formula is that it is still stochastic!

Therefore we define (could say approximate) the risk-neutral European option price as the discounted, expected payoff (or expected, discounted payoff if r_0 happens to be a stochastic interest rate),

$$F^{(\mathrm{rn})}(S(T),T)=e^{-r_0 T}\mathrm{E}^{(\mathrm{rn})}\left[\max\left[\theta\cdot\left(S^{(\mathrm{rn})}(T)-K\right),0\right]\right], \quad (7.21)$$

where $\mathrm{E}^{(\mathrm{rn})}$ denotes the expectation with respect to the corresponding risk-neutral density $\phi_{S^{(\mathrm{rn})}(t)}(s)$ or, in the abstract, with respect to a risk-neutral measure, to be determined using the Itô solution $S^{(\mathrm{rn})}(t)$ to the SDE (7.20).

• **7.1.1 Merton's (1973) Three Asset (B,S,Y), Variable Coefficient Generalization of the Black-Scholes Model, or the Black-Scholes-Merton Model:**

Merton's more general version of Black Scholes is studied for multi-dimension portfolios using an example of finance, rather than the general treatment in textbook Chapter 5.

*** Linear Stock-Price Stochastic Dynamics:**

Let $S(t)$ be the **price of stock** per share at time t, the **riskier asset**, satisfies a linear SDE:

$$dS(t)/S(t) = \mu_s(t)dt + \sigma_s(t)dW_s(t), \tag{7.22}$$

as a relative change, where the infinitesimal mean-volatility coefficients $\{\mu_s(t), \sigma_s(t)\}$ can vary in time and the diffusive differential $dW_s(t)$ is a zero-mean process with independent increments, satisfying $(dW_s)^2(t) \overset{\text{dt}}{=} dt$.

*** Linear Bond-Price Stochastic Dynamics:**

Let $B(t)$ be the **price of bond** asset at time t, in particular a default-free zero-coupon bond or discounted loan with time-to-maturity T. Then the $B(t)$ satisfies a linear diffusion SDE,

$$dB(t)/B(t) = \mu_b(t)dt + \sigma_b(t)dW_b(t), \tag{7.23}$$

where $dW_b(t)$ satisfies the same properties as $dW_s(t)$ for the stock, except for the correlation $\rho(t)$ between them, i.e., $dW_b(t)dW_s(t) \overset{\text{dt}}{=} \rho(t)dt$ is assumed, while $dW_b(t)dW_s(\tau) \overset{\text{dt}}{=} 0$ if $\tau \neq t$ **means there is no serial correlation**. If that $\sigma_b(t) < \sigma_s(t)$, then the bond is the **less risky asset** and if $\sigma_b(t) \equiv 0$ then the bond is called **risk-free or riskless**.

It can be shown that the **instantaneous correlation coefficient** between stock and bond satisfies,

$$\rho \equiv \frac{\mathrm{Cov}[dS(t), dB(t)]}{\sqrt{\mathrm{Var}[dS(t)]\mathrm{Var}[dB(t)]}} = \frac{\mathrm{Cov}[dW_s(t), dW_b(t)]}{dt}, \quad (7.24)$$

*** Instantaneous Borrowing and Shortselling is Allowed with Continuous Trading:**

Under the contract, borrowing at rate $r(t)$ from the bond is allowed to buy more stock. Shortshort selling of stock and options is also allowed with the gains saved in the bond account. Although inclusion of the bond component in the Black-Scholes model, as we have seen, was optional, but many believe that the abuse of collateral, e.g., margins placed with a broker to cover a short sale, was a significant cause of the 2007-9 economic crises, so it is a good idea to include the bond or bank account **B(t)**.

*** Option Price is a Function of Stock and Bond Prices:**
The option price per share at time t,

$$Y(t) = F(S(t), B(t), t; T, K), \tag{7.25}$$

depends on the stock $S(t)$ and bond $B(t)$ price stochastic variables, as well as on time t explicitly and parameters such as the time-to-maturity time-to-exercise T and the contracted expiration stock price K per share.

Using a two-state-dimensional version of the stochastic diffusion chain rule, the return on the option asset, initially keeping all quadratic terms in this two-dimensional Taylor expansion, is

$$\begin{aligned} dY(t) &= dF(S(t), B(t), t; T, K) \\ &\stackrel{\mathrm{dt}}{=} F_t dt + F_s dS(t) + F_b dB(t) \\ &\quad + \tfrac{1}{2}\big(F_{ss}(dS)^2(t) + 2F_{sb}dB(t)dS(t) + F_{bb}(dB)^2(t)\big), \end{aligned} \tag{7.26}$$

omitting higher order terms that are zero in dt-precision.

Here, the $\{F_s, F_b, F_{ss}, F_{sb}, F_{bb}\}$ are the set of first and second partial derivatives of $F(s,b,t;T,K)$ with respect to the underlying portfolio assets $S=s$ and $B=b$. Upon substituting for the quadratic asset differentials their leading terms of dt-precision and creating a linear dynamics for the option F,

$$dY(t) \stackrel{\text{dt}}{=} Y(t)(\mu_y(t)dt + \sigma_{ys}(t)dW_y(t) + \sigma_{yb}(t)dW_b(t)), \quad (7.27)$$

where the new coefficient are defined as

$$Y(t)\mu_y(t) \equiv \left\{\begin{matrix} F_t + \mu_s S F_s + \mu_b B F_b \\ +\frac{1}{2}\left(\sigma_s^2 S^2 F_{ss} + 2\rho\sigma_s\sigma_b SBF_{sb} + \sigma_b^2 B^2 F_{bb}\right) \end{matrix}\right\}, \quad (7.28)$$

$$Y(t)\sigma_{ys}(t) \equiv \sigma_s S F_s, \quad (7.29)$$

$$Y(t)\sigma_{yb}(t) \equiv \sigma_b B F_b. \quad (7.30)$$

* Self-Financing Portfolio Investments:

Let $N_s(t)$, $N_y(t)$ and $N_b(t)$ be the instantaneous number of shares invested in the three assets, the stock, option, and bond, at time t, respectively, such that the instantaneous values of the assets in dollars are

$$V_s(t) = N_s(t)S(t),\ V_y(t) = N_y(t)Y(t),\ \ V_b(t) = N_b(t)B(t), \tag{7.31}$$

respectively. However, it is assumed that under self-financing there is a **zero instantaneous aggregate portfolio value**,

$$V_p(t) \equiv V_s(t) + V_y(t) + V_b(t) = 0, \tag{7.32}$$

so that the bond value variable can be eliminated,

$$V_b(t) = -(V_s(t) + V_y(t)), \tag{7.33}$$

and that is the analytical reason for eliminating the bond in the Black-Scholes solution.

It is further assumed that the absolute instantaneous return from the value of the portfolio $V_p(t)$ is a linear combination of the instantaneous returns in each of the three assets, (S, Y, B), giving the **portfolio budget equation**

$$\begin{aligned} dV_p(t) &= N_s(t)dS(t) + N_y(t)dY(t) + N_b(t)dB(t) \\ &= V_s(t)\frac{dS(t)}{S(t)} + V_y(t)\frac{dY(t)}{Y(t)} + V_b(t)\frac{dB(t)}{B(t)} \end{aligned} \tag{7.34}$$

using (7.31) to convert from number of shares to asset value assuming that none of the divisors are zero. Note that the budget equation cannot be expressed as the portfolio instantaneous rate of return, since $V_p(t) = 0$ although the three assets are in return form.

Substituting for the three asset stochastic dynamics from (7.22), (7.23), (7.27) and eliminating the bond value $V_b(t)$ through (7.33), yields a more useful form of the budget equation,

$$\begin{aligned} dV_p(t) &= V_s\left(\frac{dS}{S}-\frac{dB}{B}\right)+V_y\left(\frac{dY}{Y}-\frac{dB}{B}\right) \\ &= \left((\mu_s-\mu_b)\,V_s+(\mu_y-\mu_b)\,V_y\right)dt \\ &\quad +(\sigma_s V_s+\sigma_{ys}V_y)\,dW_s(t) \\ &\quad +(-\sigma_b V_s+(\sigma_{yb}-\sigma_b)\,V_y)\,dW_b(t). \end{aligned} \tag{7.35}$$

See Merton ((1990), textbook Chapter 5; mainly an updated collection of his pioneering papers) for more justification.

Note that the first budget equation (7.34) on page L7-p49 does not really follow the Itô stochastic calculus, but states that the absolute return on the portfolio is the number of shares weighted sum of the absolute returns on the portfolio assets. However, Merton (1990) argues that the missing differential product terms, such as $dN_sS(t)$ and $dN_sdS(t)$, represent consumption or external gains to the portfolio, which would violate the self-financing assumption making the portfolio open rather than closed to just the three assets.

*** Investor Hedging the Portfolio to Eliminate Volatility.** Since many investors as individuals or as a group act to avoid stochastic effects, they tune or hedge their trading strategy, as a protection against losses, by removing volatility risk through removing the coefficients of the stock and bond fluctuations. A main purpose of the stock and bond underlying the option in the portfolio is to give sufficient flexibility to leverage or hedge the stock and bond assets to remove volatilities that would not be possible with the option alone. Hence, setting the coefficients of $dW_s(t)$ and $dW_b(t)$, respectively, to zero in (7.35),

$$\sigma_s V_s^* + \sigma_{ys} V_y^* = 0, \tag{7.36}$$

$$-\sigma_b V_s^* + (\sigma_{yb} - \sigma_b)\, V_y^* = 0. \tag{7.37}$$

The optimal system (7.36), (7.37) has a nontrivial solution for the optimal values (V_s^*, V_b^*) provided the system is singular, i.e., the determinant of the system is zero,

$$0 = \mathrm{Det}\begin{bmatrix} \sigma_s & \sigma_{ys} \\ -\sigma_b & \sigma_{yb} - \sigma_b \end{bmatrix} = \sigma_s(\sigma_{yb} - \sigma_b) + \sigma_{ys}\sigma_b, \quad (7.38)$$

which leads to the **Merton volatility fraction**

$$\frac{\sigma_{ys}}{\sigma_s} = -\frac{\sigma_{yb} - \sigma_b}{\sigma_b}, \quad (7.39)$$

provided $\sigma_s \neq 0$ and $\sigma_b \neq 0$. The single optimal option-stock value relation that makes it work is

$$V_s^* = -\frac{\sigma_{ys} V_y^*}{\sigma_s}. \quad (7.40)$$

Recalling budget constraint on V_b^*, giving

$$V_b^* = -\left(V_s^* + V_y^*\right) = -\left(1 - \frac{\sigma_{ys}}{\sigma_s}\right) V_y^*. \qquad (7.41)$$

Remarks Relating to Black-Scholes Model: *In the case of the nonstochastic, constant rate bond process, as in the more traditional Black–Scholes model,* $\mu_b = r_0$ *and* $\sigma_b = 0$*, so* $\sigma_{yb} = 0$ *and the option price is assumed to be independent of the bond price* B*, i.e.,* $F = F(S(t), t; T, K)$ *and* $F_b \equiv 0$*. Then only the optimal values (7.40) are obtained, i.e., there is no Merton volatility fraction in the traditional Black–Scholes model.*

Remarks Relating to Black-Scholes Model Continued:
However, taking the Merton volatility fraction as valid and substituting in for the definitions of the option-stock volatility σ_{ys} *and the option-bond volatility* σ_{yb} *from (7.29)–(7.30), respectively, the option price then turns out to be homogeneous [Merton (1990)] in* S *and* B,

$$Y^* = Y_s^* S + Y_b^* B. \tag{7.42}$$

Since this result is based upon the Merton volatility fraction, it does not appear in the classical Black–Scholes model, and the stock and bond dynamics no longer have common stochastic diffusion forms.

*** Zero Expected Portfolio Return:**

Further, to avoid arbitrage profits, the expected return must be zero as well. Thus, the coefficient of dt in (7.35) must be zero, aside from the assumption that $V_p(t) = 0$ would imply that $dV_p(t) = 0$, i.e.,

$$\begin{aligned} 0 &= (\mu_s - \mu_b)V_s^* + (\mu_y - \mu_b)V_y^* \\ &= \left(-(\mu_s - \mu_b)\frac{\sigma_{ys}}{\sigma_s} + (\mu_y - \mu_b)\right) V_y^*, \end{aligned} \tag{7.43}$$

assuming $V_y^* \neq 0$. Otherwise, there would be no option and no optimal values (7.40) that would follow from the Merton volatility fraction (7.39). This means that the **portfolio returns are hedged to complete equilibrium, deterministically and stochastically**.

Thus, provided the option value $V_y^* \neq 0$, by setting the coefficient of V_y^* in (7.43) to zero, **Merton's Black–Scholes fraction** becomes simply **Merton's fraction** for the expected returns, i.e.,

$$\frac{\mu_y - \mu_b}{\mu_s - \mu_b} = \frac{\sigma_{ys}}{\sigma_s}. \tag{7.44}$$

Since it does not involve either of the bond related volatilities, σ_b or σ_{yb}, this primary Merton fraction holds for the Black–Scholes model as well. The Black–Scholes fraction (7.44) states that the net drift ratio equals the option-stock volatility ratio, where the net drift is relative to the market interest/discount rate μ_b.

Summary of Lecture 7?

1. Found Distributions of Compound-Jump-Diffusions, for Simulations and Estimations, eventually.

2. Introduced Compound Interest, Continuous Time Interest, Discounting and Option Definitions.

3. Studied Black-Scholes Problem and Solution, plus (European) Put-Call Parity.

4. Explored Merton's Foundations of BS and Generalizations.

5. Next Time: Continue Merton's PDE Treatment and Perhaps Jump-Diffusion Option Pricing.