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Lecture 8 (from Chicago)

More Merton BS$^+$ Option Pricing and Jump-diffusion Financial Applications

6:30-9:30 pm, 16 November 2009 at Kent 120 in Chicago

7:30-10:30 pm, 16 November 2009 at UBS Stamford

7:30-10:30 am, 17 November 2009 at Spring in Singapore

FinM 345 Stochastic Calculus:

8. Merton BS$^+$ Option Pricing Continued and Jump-diffusion Financial Applications:

• 8.1. Merton BS$^+$ Option Pricing Continued:

- ### 8.1.1 Merton PDE of Option Pricing:

To derive the **PDE of Black–Scholes–Merton option pricing**, with definition of the option expected return μ_y in ((7.28), L70p47 or (10.24), textbook p. 296), is viewed as a PDE for the option price function with the option trajectory $Y(t)$ replaced by the composite function equivalent $F(s, b, t; T, K)$ as a function of three independent variables (s, b, t), the triplet (s, b, t) having replaced the two underlying state trajectories $(S(t), B(t))$.

This yields the PDE,

$$\begin{aligned}\mu_y F \equiv F_t + \mu_s s F_s + \mu_b b F_b \\ + 0.5\left(\sigma_s^2 s^2 F_{ss} + 2\rho\sigma_s\sigma_b s b F_{sb} + \sigma_b^2 b^2 F_{bb}\right).\end{aligned} \tag{8.1}$$

It is conceptually important to separate the view of s, b and t as three deterministic, independent PDE variables and the view of $S(t)$ and $B(t)$ as the two random SDE state trajectories in time and to use each view in the appropriate place.

Next, μ_y is eliminated using the Black–Scholes fraction [(7.44) on L7-p57 or (10.41) textbook, p. 298] with $\mu_y = \mu_b + (\mu_s - \mu_b)\sigma_{ys}/\sigma_s$ and the **option-stock induced volatility** σ_{ys} is eliminated using its definition in [(7.29), L70-p47 or (10.25) textbook, p. 296], i.e., $\sigma_{ys} = \sigma_s s F_s / F$.

The option price F can be eliminated by **Merton's homogeneous condition** [(7.42), L7-p55 or (10.38) textbook, p. 298] with y replaced by F,

$$F = sF_s + bF_b,$$

incidentally eliminating both first partials F_s and F_b, and so,

$$0 = F_t + 0.5\left(\sigma_s^2 s^2 F_{ss} + 2\rho\sigma_s\sigma_b sbF_{sb} + \sigma_b^2 b^2 F_{bb}\right). \quad (8.2)$$

This Merton **PDE of option pricing** needs side conditions, such as a final condition at the expiration time and boundary conditions in the asset variables. The PDE and conditions forming a **final value problem (FVP)**. For the FVP, the natural time variable is the time-to-maturity or time-to-exercise or time-to-go $\tau = T - t$, and $F_t = -F_\tau$.

Hence, the backward formulated PDE (8.2) in forward time t can be written as a forward diffusion or parabolic PDE in **backward time τ**,

$$F_\tau = 0.5\left(\sigma_s^2 s^2 F_{ss} v + 2\rho\sigma_s\sigma_b sbF_{sb} + \sigma_b^2 b^2 F_{bb}\right). \quad (8.3)$$

It is conceptionally important to remember that the **PDE problem**, (8.3) plus any final and boundary conditions, is a deterministic problem in realized independent variables $(s, b, t = T - \tau)$, all stochasticity being eliminated, in contrast to the **SDE problem** in the stochastic path variables $(S(t), B(t), Y(t))$, which depends on the independent variable t and underlying stochastic diffusion processes.

In the **classical Black–Scholes model**, the bond price has no volatility $\sigma_b(t)=0$, so the **Merton homogeneous result** [(7.42), L7-p55 or (10.38) textbook p. 298] does not hold since it is based upon the Merton volatility fraction, which is invalid if $\sigma_b(t)=0$. Thus, starting back at the view of the definition of μ_y as a PDE (8.1) setting all b partial derivatives to zero, but eliminating μ_y using the **Black–Scholes fraction** [(7.44), L7-p57 or (10.41) textbook p. 298] and σ_{ys} using [(7.29) on L70-p47 or (10.25) textbook, p. 296], letting the **option price function in backward time** be defined as

$$\widehat{F}(s,\tau;T,K)\equiv F(s,T-\tau;T,K),$$

which leads to Merton's **Black–Scholes option pricing PDE**, including a bond term,

$$\widehat{F}_\tau = 0.5\sigma_s^2 s^2 \widehat{F}_{ss} + \mu_b(s\widehat{F}_s + b\widehat{F}_b - \widehat{F}). \qquad (8.4)$$

If the assumption that the mean interest/discount rate is the constant market rate, $\mu_b = r_0$ along with constant stock volatility $\sigma_s = \sigma_0$, then the standard Black–Scholes option pricing PDE is obtained.

However, many texts do not use Merton's elaborate assumptions, which we have decomposed into a larger number of individual assumptions here; these texts use a different hedging argument to produce the Black–Scholes PDE and the constant rate coefficient r_0. Dropping the zero aggregate assumption, the **portfolio value** is then

$$V_P(t) = N_s(t)S(t) + N_y(t)Y(t) \tag{8.5}$$

in terms of the number of shares times the price per share for the option and the underlying stock.

Similarly, the change in the portfolio value is given by the **budget equation**

$$dV_P(t) = N_s(t)dS(t) + N_y(t)dY(t), \tag{8.6}$$

ignoring the missing differential forms as in Merton's more general version. Upon eliminating the resultant stochastic terms to form a riskless portfolio, the coefficients of $dW_s(t)$, again yields the stock-option relationship, relating the number of stock shares to that of the options

$$N_s = -N_y \widehat{F}_s, \tag{8.7}$$

called **delta hedging** since $\Delta_F \equiv \partial\widehat{F}/\partial s$ is called the **Delta** of the option [Wilmott, 2000], where the definition of σ_{ys} in ((7.29) on L70-p47 or (10.25) textbook, p. 296) has been used.

Thus,

$$V_P = N_y\,(\widehat{F} - s\widehat{F}_s),$$

where the process $Y(t)$ has been replaced by the composite function definition $Y = \widehat{F}$ in [(7.24) on L7-45 or (10.25) in textbook], and

$$dV_P = N_y\left(-\widehat{F}_\tau + 0.5s^2\widehat{F}_{ss}\right)dt.$$

Finally, it is assumed that the portfolio will earn at the riskless rate, avoiding arbitrage profits without risk,

$$dV_P(t) = r_0 V_p(t)dt, \qquad (8.8)$$

which upon eliminating V_P and dV_P leads to the **Black–Scholes option pricing PDE**,

$$\widehat{F}_\tau = 0.5\sigma_s^2 s^2\widehat{F}_{ss} + r_0(s\widehat{F}_s - \widehat{F}), \qquad (8.9)$$

independent of N_y as long as $N_y \neq 0$ and, as typically written, no longer including the bond term as in Merton's version (8.4).

The Black–Scholes option pricing equation (8.4) is a parabolic or diffusion PDE in two asset values, s and b, but degenerate in b since there is no diffusion term in b and only a drift or mean rate term $r_0 b \widehat{F}_b$.

Two elementary solutions of (8.4) can easily be verified:

- Only a stock asset: $\widehat{F}(s, b, \tau; T, K) = s$.
- Only a deterministic bond asset: $\widehat{F}(s, b, \tau; T, K) = B(T)\exp(-r_0\tau)$.

• 8.1.2 Final and Boundary Conditions for Merton PDE of Option Pricing:

In the case of the European call option, the final option price, for any value s of $S(T)$, satisfies the final option profit conditions for calls or for puts,

$$F(S(T),B(T),T;T,K)=\left\{\begin{matrix}\max[S(T)-K,0],\text{call}\\ \max[K-S(T),0],\text{put}\end{matrix}\right\} \tag{8.10}$$
$$=\max[\theta(S(T)-K),0],$$

where $\theta=1$ for calls and $\theta=-1$ for puts. Since $S(T)$ and $B(T)$ are arbitrary but nonnegative, we can replace them by the independent variables s and b respectively to form the final condition for thePDE,

$$F(s,b,T;T,K)=\max[\theta(s-K),0]. \tag{8.11}$$

We will return to the original call-put form (8.10) when transforming to new variables.

For the other boundary conditions, the discussion will be simplified to the **riskfree bond case**, i.e., $\sigma_b(t) = 0$, as assumed in the classical Black–Scholes case (8.9), except that the **time-dependent interest/discount rate,** $\mu_b(t) = r(t)$, will be retained. In the case of risky bonds, the boundary conditions are given by diffusion PDEs instead of explicit functions or values, so solving the PDE (8.3) by computational methods, as in Chapter 8, is more practical.

The number of boundary conditions depends on the highest order partial derivative for each independent state variable in the PDE, one condition if it is first order and two conditions if it is second order.

Thus, for (8.3) it is **two boundary conditions in the stock and one in the bond**. Time is not a state variable, but there is one final condition (technically an initial condition for the backward time variable τ) since the time derivative is first order.

At the **zero stock price**, $s=0$, Merton's Black–Scholes PDE (8.4) reduces to

$$\widehat{F}_\tau(0,b,\tau;T,K) = \widehat{r}(\tau)\left(b\widehat{F}_b - \widehat{F}\right) \tag{8.12}$$

upon setting s to zero in the coefficients, where $\widehat{r}(\tau) \equiv r(T-\tau)$ and assuming the derivatives are bounded, which is a risky assumption before finding the solution. This is a first order PDE, all of which are classified as hyperbolic PDEs, and the usual method of constructing a solution is called the **method of characteristics** [Sneddon (1957].

Noting that the PDE problem is a deterministic problem, the PDE (8.12) is compared to the **deterministic (non-Itô!) chain rule** for $\widetilde{F}(b,\tau)\equiv\widehat{F}(0,b,\tau;T,K)$,

$$d\widetilde{F} = \widetilde{F}_\tau d\tau + \widetilde{F}_b db, \tag{8.13}$$

assuming that the differentials $d\tau$ and db can be varied independently, and the ODEs for the **characteristic path** are written maintaining relative proportions between the differentials of (8.13) and the corresponding coefficients of (8.12),

$$\frac{d\tau}{1} = -\frac{db}{\widehat{r}(\tau)b} = -\frac{d\widetilde{F}}{\widehat{r}(\tau)\widetilde{F}}.$$

Solving these ODEs successively in pairs,

$$b = \widetilde{B}(\tau) = \kappa e^{-R(\tau)}, \tag{8.14}$$

where κ is a characteristic path constant of integration and the cumulative rate for time-dependent $r(t)$ is

$$R(\tau) \equiv \int_0^\tau \widehat{r}(q)dq \equiv \overline{\widehat{r}}(0,\tau)\tau = \int_0^\tau r(T-q)dq \equiv \overline{r}(T-\tau,T)\tau, \tag{8.15}$$

so averages like $\overline{r}(t,t+\Delta t) \equiv \int_t^{t+\Delta t} r(q)dq/\Delta t$ replace constants like r_0 in the variable coefficients case, and

$$\widetilde{F} = f(\kappa)e^{-R(\tau)},$$

where $f = f(\kappa)$ is an arbitrary function of integration depending on the constant κ from the (8.14) integration. Using the first integral (8.14) to eliminate κ in favor of $\widetilde{B}$ and τ yields

$$\widetilde{F}(\widetilde{B}(\tau),\tau) = f\left(\widetilde{B}(\tau)e^{R(\tau)}\right)e^{-R(\tau)}. \tag{8.16}$$

It is not necessary to know much about the method of characteristics, since the reader can verify the solution by the usual substitution procedure. The arbitrary function f can be eliminated by applying the final condition (8.11) at $\tau = 0$ with $R(0) = 0$,

$$\begin{aligned}\widetilde{F}(\widetilde{B}(0),0) = f(\widetilde{B}(0)) = F(0,\widetilde{B}(0),T;T,K)\\ = \max[\theta(-K),0] = 0.5(1-\theta)K,\end{aligned}$$

i.e., non-zero only for a put when $\theta = -1$. Since $\widetilde{B}(0) = B(T)$ is considered arbitrary at this point, $f(\widetilde{B}) = 0.5(1-\theta)K$, a constant (**beware**: Merton (1973) assumes $B(T) = 1$), leading to the **complete particular solution**

$$\widetilde{F}(b,\tau) = \widehat{F}(0,b,\tau;T,K) = 0.5(1-\theta)Ke^{-R(\tau)}, \quad (8.17)$$

independent of $b = \widetilde{B}(\tau)$.

Note that $\widetilde{B}(\tau)$ is a deterministic path function of a deterministic **ODE problem** since it is derived from the deterministic PDE problem, (8.12) plus conditions, so is different from the stochastic path function $\widehat{B}(\tau)$ for the SDE problem, or more precisely the stochastic ODE problem. The boundary condition (8.17) corresponds to a boundary condition used by Wilmott (2000) for finite differences applied to Black–Scholes-type models.

However, since we cannot assume the partial derivatives are bounded for the full Merton model (8.3), we will only assume that the option price will be bounded in the limit of zero stock price:

$$\widehat{F}(s, b, \tau; T, K) \text{ is bounded as } s \to 0^+. \tag{8.18}$$

For large s, it is more difficult to find the proper boundary condition. ***However, one heuristic choice is to assume that for large s the diffusion term will be exponentially small so the drift terms will dominate:***

$$\widehat{F}_\tau \simeq \widehat{r}(\tau)(s\widehat{F}_s + b\widehat{F}_b - \widehat{F}). \qquad (8.19)$$

As with the small stock price limit, the conjecture (8.19) needs to be verified for a solution. Again applying the method of characteristics to $\widetilde{F}(s,b,\tau) \equiv \widehat{F}(s,b,\tau;T,K)$, or checking by substitution, but with four variables, $\left\{\tau, b, s, \widehat{F}\right\}$, instead of three,

$$\frac{d\tau}{1} = -\frac{db}{\widehat{r}(\tau)b} = -\frac{ds}{\widehat{r}(\tau)s} = -\frac{d\widehat{F}}{\widehat{r}(\tau)\widehat{F}}.$$

Integration leads to three constants or functions of integration.

Two of the functions of integration can be eliminated in favor of the independent variables s and b,

$$\widehat{F}(S, B, \tau; T, K) = g\left(Se^{R(\tau)}, Be^{R(\tau)}\right) e^{-R(\tau)}, \quad (8.20)$$

where $g = g(s\exp(R(\tau)), b\exp(R(\tau)))$ is an arbitrary function of integration obtained by integrating both the stock and bond characteristic ODEs effectively generating two constants of integration, and $R(\tau)$ is given in (8.15).

Applying the final condition (8.11) when $s > K$ yields

$$\widehat{F}(s, b, 0; T, K) = \max[\theta(s-K), 0] = 0.5(1+\theta)(s-K),$$

so that g is a constant function and the complete particular solution

$$\widehat{F}(S, B, \tau; T, K) \simeq 0.5(1 + \theta)(S - Ke^{-R(\tau)}). \quad (8.21)$$

A similar boundary condition is also specified in Wilmott's (2000) finite difference applications. However, it turns out we will not need this condition here, but the condition suggests that the option price will not be bounded as $s \longrightarrow +\infty$.

The bond boundary condition or conditions are not as straightforward, since the final bond price per share does not appear explicitly in the final option profit formula. At the zero bond price, $b = 0$, the Black–Scholes PDE (8.9) reduces to

$$\widehat{F}_\tau(s, 0, \tau; T, E) = \widehat{r}(\tau)(s\widehat{F}_s - v\widehat{F}) + 0.5\sigma_s^2 s^2 \widehat{F}_{ss} \qquad (8.22)$$

upon setting b to zero in the coefficients, assuming the derivatives are bounded.

However, (8.22) is a diffusion equation rather than a boundary value, so there has been very little simplification of the original Black–Scholes PDE except that the dimension has been reduced to one from two state variables. This may still be useful for computational methods. The reduction in dimension is similar for the Merton version (8.3) of the Black–Scholes option pricing PDE, the only difference being that the drift term is absent. For either PDE, setting $b = B(T)$ in the PDE leads to no simplification since $B(T)$ would be arbitrary. There is still hope, since Merton has a way of transforming away $B(T)$ analytically, but this transformation is modified here.

- ## 8.1.3 Black-Scholes$^+$ European Option Pricing Formula by Risk-Neutrality (RN):

The lectures will not continue with the rest of Merton's derivation of the BS pricing formulas, since they are mainly of interest in ***academic finance****, although there are some good examples of applied analytical techniques.* From [(7.21) on L7-p42)], the RN prices are

$$F^{(\mathrm{rn})}(S(T),T)=e^{-r_0 T}\mathrm{E}^{(\mathrm{rn})}\Big[\max\Big[\theta\cdot\Big(S^{(\mathrm{rn})}(T)-K\Big),0\Big]\Big], \quad (8.23)$$

at the exercise time with implied conditioning on $S(0) = S_0$, so generalizing with conditioning on any $t\in[0,T)$, then

$$F^{(\mathrm{rn})}(s,t)=e^{-r_0(T-t)}\mathrm{E}^{(\mathrm{rn})}\Big[\max\Big[\theta\cdot\Big(S^{(\mathrm{rn})}(T)-K\Big),0\Big] \Big| S(t)=s\Big], \quad (8.24)$$

recalling that $\theta=1$ of a call and $\theta=-1$ for a put. Further by stationarity of $W(t)$, we have

$$\Phi_{W(T)-W(t)}(w) \overset{\mathrm{dist}}{=} \Phi_{W(T-t)}(w) = \Phi_n(w,0,T-t).$$

So the we can write the more useful and applicable risk-neutral stock price solution at exercise relative to current time t and current state $S(t)=s$, with replacement $\mu_0=r_0$, from [(3.6) on L3-p58], as

$$S^{(\mathrm{rn})}(T)=se^{(r_0-0.5\sigma_0^2)(T-t)+\sigma_0 W(T-t)} \quad (8.25)$$

or the log-return log-normally distributed form,

$$\ln(S^{(\mathrm{rn})}(T))=\ln(s)+(r_0-0.5\sigma_0^2)(T-t)+\sigma_0 W(T-t). \quad (8.26)$$

For notational simplicity and for later generalizations, let

$$\mu_\ell=\mu_\ell(\tau)\equiv(r_0-0.5\sigma_0^2)\tau = \mathrm{E}[\ln(S^{(\mathrm{rn})}(T)/s)|S(t)=s],$$

$$\sigma_\ell^2=\sigma_\ell^2(\tau)\equiv\sigma_0^2\tau = \mathrm{Var}[\ln(S^{(\mathrm{rn})}(T)/s)|S(t)=s].$$

With moderately more work, we can generalize this notation to incomplete markets like jump-diffusion markets.

Given $S(t)=s$ conditioning, let $\phi_{S^{(\mathrm{rn})}(T)}(y)$ be the risk-neutral density for $S^{(\mathrm{rn})}(T)$ with critical of the maximum ramp function at $y^*=K$ and let the standard normal variable be $z=(\ln(y/s)-\mu_\ell)/\sigma_\ell=w/\sqrt{\tau}$, having critical value at $z^*(s,\tau)=(\ln(K/s)-\mu_\ell)/\sigma_\ell^2(\tau)$, with change of densities, $\phi_{S^{(\mathrm{rn})}(T)}(y)dy=\phi_{S^{(\mathrm{rn})}(T)}(y)(dy/dz)dz=\phi_{W(T-t)/\sqrt{T-t}}(z)dz$, with last term just $\phi_n(z;0,1)dz$. Hence,

$$
\begin{aligned}
F^{(\mathrm{rn})}(s,t) &= e^{-r_0(T-t)}\int_{-\infty}^{+\infty}\max[\theta(y-K),0]\phi_{S^{(\mathrm{rn})}(T)}(y)dy \\
&= e^{-r_0(T-t)}\theta\int_{K}^{\theta\cdot\infty}(y-K)\phi_{S^{(\mathrm{rn})}(T)}(y)dy \\
&= e^{-r_0(T-t)}\theta\int_{z^*(s,\tau)}^{\theta\cdot\infty}\left(se^{\sigma_\ell z+\mu_\ell}-K\right)\phi_n(z;0,1)dz \\
&= se^{-0.5\sigma_\ell^2}\theta\int_{z^*(s,\tau)}^{\theta\cdot\infty}e^{\sigma_\ell z}\phi_n(z;0,1)dz \\
&\quad -Ke^{-r_0(T-t)}\theta\int_{z^*(s,\tau)}^{\theta\cdot\infty}\phi_n(z)dz.
\end{aligned}
$$

The final integrals are called the **tail probabilities**. We can get the BS formulas, with some normal distribution manipulations that we have done before, such as

$$\theta\int_{z^*(s,\tau)}^{\theta\cdot\infty}\phi_n(z;0,1)dz=\delta_{\theta,1}-\theta\Phi_n(z^*;0,1)=\Phi_n(-\theta z^*;0,1),$$

$$\theta\int_{z^*(s,\tau)}^{\theta\cdot\infty}e^{\sigma_\ell z}\phi_n(z;0,1)dz\overset{\text{cts}}{=}\frac{1}{\sqrt{2\pi}}e^{+0.5\sigma_\ell^2}\theta\int_{z^*(s,\tau)}^{\theta\cdot\infty}e^{-0.5(z-\sigma_\ell)^2}dz$$

$$=e^{+0.5\sigma_\ell^2}\Phi_n(-\theta(z^*-\sigma_\ell);0,1),$$

the last lines by the completing the square technique. Thus,

$$F^{(\mathrm{rn})}(s,t)=s\Phi_n(\theta(\sigma_\ell-z^*);0,1)-Ke^{-r_0\tau}\Phi_n(-\theta z^*;0,1).$$

When the normal distribution arguments are replaced by the beginning notational definitions, then,

$$\begin{aligned}\theta(\sigma_\ell-z^*(s,\tau))&=\theta(\ln(s/K)+\mu_\ell+\sigma_\ell^2)/\sigma_\ell\\&=\theta(\ln(s/K)+(r_0+\sigma_0^2)\tau/(\sigma_0\sqrt{\tau})\\&=\theta d_1(s,\tau;K,r_0,\sigma_0).\end{aligned}$$

The d_1-function is the first normal argument function of the Black-Scholes formula and the second, d_2 is found next,

$$\begin{aligned} -\theta z^*(s,\tau) &= \theta(\ln(s/K)+\mu_\ell)/\sigma_\ell \\ &= \theta(\ln(s/K)+(r_0-\sigma_0^2)\tau)/(\sigma_0\sqrt{\tau}) \\ &= \theta d_2(s,\tau;K,r_0,\sigma_0) \\ &= \theta(d_1(s,\tau;K,r_0,\sigma_0)-\sigma_0\sqrt{\tau}). \end{aligned}$$

yielding:

Theorem 8.1. Black-Scholes European Option Pricing Formula:

$$\begin{aligned} F^{(\mathrm{rn})}(s,t) = {} & s\Phi_n(\theta d_1(s,\tau;K,r_0,\sigma_0);0,1) \\ & -Ke^{-r_0\tau}\Phi_n(\theta d_2(s,\tau;K,r_0,\sigma_0);0,1), \end{aligned} \tag{8.27}$$

with the ***BS European option prices combined with*** $\theta=1$ ***for the call and*** $\theta=-1$ ***for the put****, but usually the standard normal notation* $\Phi(d_i(s,\tau))=\Phi_n(d_i(s,\tau);0,1)$ *is used, where recall* $\tau=T-t$*, the time-to-exercise.*

- ## 8.1.3 Jump-Diffusion (JD) European Option Pricing Formula by Risk-Neutrality (RN) and EMM:

First consider the partially constant coefficient compound or mark-jump-diffusion asset or stock price model,

$$dS(t)/S(t) = \mu_0 dt + \sigma_0 dW(t) + d\mathrm{CP}(t,Q), \quad (8.28)$$

with $S(0) = S_0$ and $\mathrm{CP}(t,Q) = \sum_{j=1}^{P(t)} \nu(Q_j)$. However, we will be interested in the conditioned value at the current time, $S(t) = s$ for $s > 0$ with $\overline{\nu} \equiv \mathrm{E}[\nu(Q)]$, such that

$$\mathrm{E}[dS(t)|S(t)=s] = s(\mu_0 + \lambda_0 \overline{\nu})dt,$$

$$\mathrm{Var}[dS(t)|S(t)=s] = s(\sigma_0^2 + \lambda_0 \overline{\nu^2})dt,$$

so that in a risk-neutral jump-diffusion environment, we need that the earning rate is at the risk-free rate r_0, so

$$\mu_0^{(\mathrm{rn})} + \lambda_0 \overline{\nu} = r_0, \quad \text{or} \quad \mu_0^{(\mathrm{rn})} = r_0 - \lambda_0 \overline{\nu},$$

taking μ_0 as the eliminant.

Hence, the **risk-neutral jump-diffusion stock price SDE** is

$$dS^{(\mathrm{rn})}(t) = S^{(\mathrm{rn})}(t)\left(r_0dt + \sigma_0dW(t) + d\widetilde{\mathrm{CP}}(t,Q)\right), \quad (8.29)$$

where $\widetilde{\mathrm{CP}}(t,Q) \equiv \sum_{j=1}^{P(t)} \nu(Q_j) - \lambda_0\overline{\nu}t$ is the mean-zero compound Poisson.

Incidentally, **you** can show that $\widetilde{\mathrm{CP}}(t,Q)$, properly constructed, is a **martingale**, while **W(t)** for diffusion is started as one. In the **"abstract"**, we have given the Poisson process an **equivalent martingale measure (EMM) shift of its drift**, i.e., $d\mathrm{CP}(t,Q) = d\widetilde{\mathrm{CP}}(t,Q) + \lambda_0\overline{\nu}dt$. In general, we can also do something similar for the diffusion by letting $dW(t) = d\widetilde{W}(t) + \gamma_0dt$ for some constant $\gamma_0 \neq 0$, so

$$\begin{aligned} dS(t)/S(t) &= (\mu_0 + \gamma_0\sigma_0 + \lambda_0\overline{\nu})dt + \sigma_0d\widetilde{W}(t) + d\widetilde{\mathrm{CP}}(t,Q) \\ &\overset{emm}{=} r_0dt + \sigma_0d\widetilde{W}(t) + d\widetilde{\mathrm{CP}}(t,Q). \end{aligned}$$

Risk-neutrality has been enforced by selecting

$$\gamma_0 \overset{emm}{=} \frac{r_0 - \mu_0 - \lambda_0 \overline{\nu}}{\sigma_0}$$

and γ_0 is called the **jump-diffusion risk-premium**, while its components are the **diffusion risk-premium**

$$\gamma_0^{(\mathrm{d})} \overset{emm}{=} \frac{r_0 - \mu_0}{\sigma_0}$$

and the additive **jump risk-premium**

$$\gamma_0^{(\mathrm{j})} \overset{emm}{=} \frac{-\lambda_0 \overline{\nu}}{\sigma_0}$$

weighted by the diffusive volatility by convention. One can show that using the solution [(4.39) on L4-p51],

$$\mathbf{E}[e^{-(\mu_0+\lambda_0\overline{\nu})t}S(T)|S(t)=s]=\mathbf{E}[e^{-(r_0+\gamma_0\sigma_0)t}S(T)|S(t)=s]=s,$$

so the both arguments of the expectations are martingales.

If one wanted to, the equivalent martingale measure $\widetilde{M}$ can be calculated using a **Girsanov change of measure** with the **Radon-Nykodym derivative**, in the case of **pure diffusions (D)** [textbook, p. 383],

$$\frac{d\widetilde{M}^{(\mathrm{d})}(T)}{dM^{(\mathrm{d})}(t)} = \frac{\phi_{\widetilde{W}(T)}(\widetilde{w})d\widetilde{w}}{\phi_{W(t)}(w)dw} = e^{+\gamma_0^{(\mathrm{d})}\left(\widetilde{w} - \gamma_0^{(\mathrm{d})}t/2\right)},$$

while a “concrete” jump-diffusion (JD) version of Girsanov’s theorem can be found in [textbook, Chapt. 12, p. 384ff].

Risk-Neutral Assumptions Summary from Merton's Jump-Diffusion Paper (1976):

- Jumps are due to **Extreme Changes in Firm's Specifics**, i.e., ***Non-Systematic Risks***, e.g., bankruptcy, adverse legal rulings, unfavorable publicity, important discoveries, etc.
- ***Portfolio-Market Return Correlation beta*** (i.e., $\mathrm{Cov}[R_S, R_M]/\mathrm{Var}[R_M]$, where return $R_X = \Delta X/X$ for $X = S$ or M (market reference)) is **Zero** and can be constructed by ***Delta*** $= \partial V_p/\partial s$ ***Hedging***.
- Thus, ***Jump-Diffusion Model is Arbitrage-Free***.
- $\therefore$ ***Risk-Neutral World*** (a Hull-ism) $\Longrightarrow \mathrm{E}[S(t)] = S_0 \exp(r_0 t) \Longrightarrow \mu_0 + \lambda_0 \bar{\nu} = r \Longrightarrow \mu_0 = \mu^{(\mathrm{rn})} \equiv r_0 - \lambda_0 \bar{\nu}$.
  - Similarly, for time-dependent coefficients,

$$\mu(t) = \mu^{(\mathrm{rn})}(t) \equiv r_0 - \lambda \mathrm{E}[\nu(t, Q)].$$

Returning to the main task of **calculating the jump-diffusion risk-neutral European option pricing problem constant with coefficients**, using the solution [(6.22) on L6-p21 or (5.51) textbook, p. 144], with $\nu(Q) = e^{Q} - 1$ but in time-shifted risk-neutral form by $\tau = T - t$ and $S^{(\mathrm{rn})}(t) = s$,

$$S^{(\mathrm{rn})}(T) = s\exp\left((r_0 - \lambda_0\overline{\nu} - \sigma_0^2/2)\tau + \sigma_0 W(\tau) + \sum_{j=1}^{P(\tau)} Q_j\right),$$

Next using iterated expectations and the law of total probability let the risk-neutral options price,

$$\begin{aligned} F^{(\mathrm{rn})}(s,t) &= e^{-r_0\tau}\mathrm{E}^{(\mathrm{rn})}\left[\max\left[\theta\left(S^{(\mathrm{rn})}(T) - K\right), 0\right] \middle| S(t) = s\right] \\ &= e^{-r_0\tau}\mathrm{E}_{W(\tau)}\left[\mathrm{E}_{P(\tau)}\left[\mathrm{E}_{Q}\left[\max\left[\theta(S^{(\mathrm{rn})}(T) - K, 0\right]\right.\right.\right. \\ &\quad \left.\left.\left. |P(\tau), S^{(\mathrm{rn})}(t) = s\right]\right]\right] \\ &= e^{-r_0\tau}\textstyle\sum_{k=0}^{\infty} p_k(\lambda_0\tau)\mathrm{E}_{\mathcal{S}_k}\left[\theta\displaystyle\int_{w^*(s,\tau,\mathcal{S}_k)}^{\theta\infty} dw\,\phi_{W(\tau)}(w)\right. \\ &\quad \left.\left(se^{(r_0 - \lambda_0\overline{\nu} - \sigma_0^2/2)\tau + \sigma_0 w + \mathcal{S}_k} - K\right)\right]. \end{aligned}$$

Here, the partial sum $\mathcal{S}_k \equiv \sum_{j=1}^{k} Q_j$ and the w-critical value such that $S^{(\mathrm{rn})}(T) = K$ is, in standard normal form,

$$z^*(s,\tau,\mathcal{S}_k) = \frac{w^*(s,\tau,\mathcal{S}_k)}{\sqrt{\tau}} = \frac{\ln(K/s) - (r_0 - \lambda_0\overline{\nu} - \sigma_0^2/2)\tau - \mathcal{S}_k}{\sigma_0\sqrt{\tau}}.$$

You can show by IID properties that the basic statistics of $\mathcal{S}_k$ $\mathrm{E}[\mathcal{S}_k] = k\mathrm{E}[Q] = k\overline{Q} = k\mu_j$ and $\mathrm{Var}[\mathcal{S}_k] = k\mathrm{Var}[Q] = k\sigma_j^2$ for $k \geq 0$. Reformulating the option price, let

$$F^{(\mathrm{rn})}(s,t) = \sum_{k=0}^{\infty} p_k(\lambda_0\tau)\mathrm{E}_{\mathcal{S}_k}[A(s,\tau,\mathcal{S}_k) - B(s,\tau,\mathcal{S}_k)]$$

where the **tail probability functions** are

$$\begin{aligned} A(s,\tau,\mathcal{S}_k) &= s e^{\mathcal{S}_k - (\lambda_0\overline{\nu} + \sigma_0^2/2)\tau}\theta\int_{z^*(s,\tau,\mathcal{S}_k)}^{\theta\infty} dz\, \phi_n(z;0,1)e^{\sigma_0\sqrt{\tau}z} \\ &\overset{\mathrm{cts}}{=} s e^{\mathcal{S}_k - \lambda_0\overline{\nu}\tau}\Phi_n(\theta(\sigma_\ell - z^*(s,\tau,\mathcal{S}_k));0,1) \\ &= s e^{\mathcal{S}_k - \lambda_0\overline{\nu}\tau}\Phi_n(\theta\, d_1(s\exp(\mathcal{S}_k - \lambda_0\overline{\nu}\tau),\tau);0,1), \end{aligned}$$

where recall $\sigma_\ell^2 = \sigma_0^2\tau$, and that $d_1(s,\tau)$ is the first BS argument function.

Similarly, where

$$B(s,\tau,\mathcal{S}_k) = Ke^{-r_0\tau}\theta\int_{z^*(s,\tau,\mathcal{S}_k)}^{\theta\infty} dz\,\phi_n(z;0,1)$$

$$\overset{\text{cts}}{=} Ke^{-r_0\tau}\Phi_n(-\theta z^*(s,\tau,\mathcal{S}_k);0,1)$$

$$= Ke^{-r_0\tau}\Phi_n(\theta\, d_2(s\exp(\mathcal{S}_k-\lambda_0\overline{\nu}\tau),\tau);0,1),$$

where $d_2(s,\tau)$ is the second BS normal argument function. Relabeling the Black-Scholes option price version from (8.27),

$$F^{(\mathrm{bs})}(s,\tau;K,r_0,\sigma_0) = s\Phi_n(\theta d_1(s,\tau;K,r_0,\sigma_0);0,1)$$

$$-Ke^{-r_0\tau}\Phi_n(\theta d_2(s,\tau;K,r_0,\sigma_0);0,1),$$

so the jump diffusion formula version can be written as a mixture of BS option prices and formulated as a theorem:

Theorem 8.2. Jump-Diffusion Risk-Neutral European Option Pricing Formula:

$$F^{(\mathrm{rn})}(s,t) = \sum_{k=0}^{\infty} p_k(\lambda_0\tau)\mathrm{E}_{\mathcal{S}_k}\Big[F^{(\mathrm{bs})}(s\exp(\mathcal{S}_k-\lambda_0\overline{\nu}\tau),\tau;*)\Big]. \quad (8.30)$$

Remarks:

- *The premium is the initial option price which is* $F^{(\mathrm{rn})}(s,0)$ *with the time-to exercise* $\tau = T$. *However, the general formulation means that* $F^{(\mathrm{rn})}(s,t)$ *is the premium for an option starting at time* t *for an exercise time of* τ *maturing at* T.
- *The option prices, generally, depend on an infinite number of Black-Scholes options, averaged as Poisson counting sums and a corresponding sum of jump-amplitude marks. The* $k = 0$ *term, when* $\mathcal{S}_0 = 0$, *is the pure Black-Scholes result with an extra jump discount,*
$$F_0^{(\mathrm{rn})}(s,t) = \exp(-\lambda_0\tau)F^{(\mathrm{bs})}(s\exp(-\lambda_0\overline{\nu}\tau),\tau;*).$$
- *Option maturities usually are in months or a few years, so the zero-one jump law is not useful in truncating the Poisson sum at some low jump count of* k.
- *Unlike the mean and variance of* $\mathcal{S}_k$, *the expectation* $\mathrm{E}_{\mathcal{S}_k}$ *of the* k*th shifted Black-Scholes option price term, the mark densty* $\phi_Q(q)$ *needs to be known.*

- ## 8.1.4 Monte Carlo Simulated European Option Pricing for Log-Uniform Jump-Diffusions:

Merton in 1976 first gave the jump-diffusion European option pricing formula as a Poisson sum of Black-Scholes option prices using the thin-tailed log-normal jump-amplitude distribution, but the derivation details (Merton's works are usually short on details) given in the last section was from a ***Monte Carlo option pricing paper*** of Zhu and Hanson (2005) using the fat-tailed log-uniform jump-amplitude distribution.

If the mark density $\phi_Q(q)$ for the log-return is uniform on **(a,b)** than so is that for $\mathcal{S}_1 = Q_1$, but even for $k = 2$, as previous noted and shown in the textbook, $\mathcal{S}_2 = Q_1 + Q_2$ has a triangular distribution on $(2a, 2b)$. The partial sum densities of $\mathcal{S}_k$ become more complex with k.

However, the simulation of the European call option pricing at $t = T$ and $s = S_0$ can be simplified by not simulating $\mathcal{S}_k$ for each k, but by simulating with the compound process itself,

$$\widehat{\mathcal{S}}(T) = \sum_{j=1}^{P(T)} Q_j, \qquad (8.31)$$

while letting $\theta = 1$ for the call, $\widehat{\mathcal{C}}^{(\mathrm{rn})} = F^{(\mathrm{rn})}$ and $\mathcal{C}^{(\mathrm{bs})} = F^{(\mathrm{bs})}$, so our more compact and reassembled compound Poisson expectation becomes

$$\widehat{\mathcal{C}}^{(\mathrm{rn})}(s, T) = \mathrm{E}_{\widehat{\mathcal{S}}(T)}\left[\mathcal{C}^{(\mathrm{bs})}\left(se^{\widehat{\mathcal{S}}(T) - \lambda_0 \overline{\nu} T}, T\right)\right]. \qquad (8.32)$$

Consider $i=1:n$ **Poisson counter samples** P_i from $P(T)$, thus the samples are IID. Then let the $U_{i,j}$ for $j=1:P_i$ be **IID standard uniform variates**, i.e., on $(0,1)$, for each i such that the log-jump-amplitudes on (a,b) are

$$Q_{i,j}=a+(b-a)U_{i,j}$$

and

$$\widehat{\mathcal{S}}_i=\sum_{j=1}^{P_i}Q_j=aP_i+(b-a)\sum_{j=1}^{P_i}U_{i,j}, \qquad (8.33)$$

for $i=1:n$ IID compound Poisson random variables with corresponding uniform jump-amplitudes.

The **simple Monte Carlo estmate** (see Hanson [(2007), Chapter 9] for an introduction or Glasserman (2004) for the main Monte Carlo reference for finance; Monte Carlo is named for the gambling capital of Europe, but it originated and was implemented at Los Alamos, so it is often called the **Metropolis algorithm** for the implementer, with Fermi, Ulam and von Neumann playing important roles) is based upon the **average approximation to an integral** of interest, here to finance,

$$\widehat{\mathcal{C}}_n = \frac{1}{n}\sum_{i=1}^{n}\mathcal{C}^{(\mathrm{bs})}\left(s e^{\widehat{\mathcal{S}}_i - \lambda_0 \overline{\nu}\, T}, T\right) \equiv \frac{1}{n}\sum_{i=1}^{n}\widehat{\mathcal{C}}_i^{(\mathrm{bs})},$$

noting that the **Black-Scholes samples** $\widehat{\mathcal{C}}_i^{(\mathrm{bs})}$ are IID random variables based upon the **compound Poisson samples** $\widehat{\mathcal{S}}_i$.

For the asymptotic limits, the **strong law of large numbers (SLLN)** implies $\widehat{\mathcal{C}}_n \longrightarrow \mathcal{C}(s,T)$ with probability one as $n \longrightarrow \infty$. Concerning the convergence error, using the IID property of $\widehat{\mathcal{C}}_i^{(\mathrm{bs})}$, the standard deviation is given by

$$\sigma_{\widehat{\mathcal{C}}_n} = \frac{\widehat{\sigma}^{(\mathrm{bs})}}{\sqrt{n}} \equiv \sqrt{\frac{\mathrm{Var}\left[\mathcal{C}^{(\mathrm{bs})}\left(se^{\widehat{S}_i(T)-\lambda\overline{\nu}T}, T\right)\right]}{n}} \equiv \sqrt{\frac{\mathrm{Var}\left[\widehat{\mathcal{C}}_i^{(\mathrm{bs})}\right]}{n}}.$$

However, this is too difficult to calculate, but may be estimated by the **unbiased sample variance**,

$$\widehat{\mathrm{s}}^{(\mathrm{bs})} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(\widehat{\mathcal{C}}_i^{(\mathrm{bs})} - \widehat{\mathcal{C}}_n\right)^2}.$$

Note that due to the $\mathrm{O}(1/\sqrt{n})$, in order to reduce the standard deviation $\sigma_{\widehat{\mathcal{C}}_n}$ by a factor of ten, the number of simulations n has to be increased one hundredfold.

Thus, for reasonable accuracy, a very large number of samples are needed for the simulation. However, there are many modifications of the simple Monte Carlo techniques with the goal to reduce the size of the variance $(\sigma^{(\mathrm{bs})})^2$. These **variance reduction techniques** include **thetic-antithetic (AT)** (i.e., the thesis and its opposite) techniques and **optimal control variate (OCV) techniques**. Let $\widehat{\mathcal{S}}_i^{(a)}$ and $\widehat{\mathcal{C}}_i^{(abs)}$ be the antithetic variates to $\widehat{\mathcal{S}}_i$ and $\widehat{\mathcal{C}}_i^{(\mathrm{bs})}$, respectively, then the thetic-antithetic averaged, BS discounted payoff be

$$X_i = 0.5\left(\widehat{\mathcal{C}}_i^{(\mathrm{bs})} + \widehat{\mathcal{C}}_i^{(abs)}\right), \tag{8.34}$$

where the antithetic is

$$\widehat{\mathcal{C}}_i^{(abs)} \equiv \mathcal{C}^{(\mathrm{bs})}(S_0 e^{\widehat{\mathcal{S}}_i^{(a)} - \lambda\overline{\nu}T}, T)$$

for $i = 1:n$.

Similarly, **thetic-antithetic averaged jump-amplitude partial sum exponential** is

$$\boldsymbol{Y_i = 0.5\left(\exp\left(\widehat{\mathcal{S}}_i\right) + \exp\left(\widehat{\mathcal{S}}_i^{(a)}\right)\right)}\ ,$$

So the antithetic and thetic variates can be used together to double the sample size without significant computational cost [Phelim Boyle (1977), father of Monte Carlo options]. The $\boldsymbol{Y_i}$ are also used in the **control deviation** of the **control adjusted payoff**

$$Z_i(\alpha) = X_i - \alpha \cdot (Y_i - \exp(\lambda_0 \overline{\nu} T))\ , \tag{8.35}$$

where α is the **actual adjustable control parameter**.

The sample mean of $Z_i(\alpha)$ produces the ***Monte Carlo estimator*** for $\mathcal{C}(S_0, T)$, since

$$\overline{Z}_n(\alpha) \equiv \sum_{i=1}^{n} Z_i(\alpha)/n = \overline{X}_n - \alpha(\overline{Y}_n - \exp(\lambda_0 \overline{\nu} T)),$$

is an unbiased estimation with $\mathrm{E}[\overline{Z}_n(\alpha)] = \mathcal{C}(S_0, T)$ using IID mean properties $\mathrm{E}[\overline{X}_n] = \mathrm{E}[X_i] = \mathcal{C}(S_0, T)$ and $\mathrm{E}[\overline{Y}_n] = \mathrm{E}[Y_i] = \exp(\lambda_0 \overline{\nu} T)$. The variance of $\overline{Z}_n(\alpha)$ is

$$\sigma^2_{\overline{Z}_n(\alpha)} \equiv \mathrm{Var}\left[\overline{Z}_n(\alpha)\right] = \mathrm{Var}[Z_i(\alpha)]/n\ ,$$

following from IID property of the $Z_i(\alpha)$. However,

$$\mathrm{Var}[Z_i(\alpha)] = \mathrm{Var}[X_i] - 2\alpha \mathrm{Cov}[X_i, Y_i] + \alpha^2 \mathrm{Var}[Y_i].$$

So, the **optimal parameter** α^* to minimize $\mathrm{Var}[Z_i(\alpha)]$ is

$$\alpha^* = \mathrm{Cov}[X_i, Y_i]/\mathrm{Var}[Y_i], \tag{8.36}$$

i.e., related to the BS (X_i) and jump-amplitude (Y_i) averaged antithetic-thetic variates.

Using this optimal parameter α^*,

$$\mathrm{Var}[Z_i^*] \equiv \mathrm{Var}[Z_i(\alpha^*)] = \left(1 - \rho^2_{X_i,Y_i}\right)\mathrm{Var}[X_i],$$

where ρ_{X_i,Y_i} is the correlation coefficient between X_i and Y_i. We also know that

$$\mathrm{Var}[X_i] = 0.5\left(1 + \rho_{\widehat{\mathcal{C}}_i^{(bs)},\widehat{\mathcal{C}}_i^{(abs)}}\right)\mathrm{Var}\left[\widehat{\mathcal{C}}_i^{(bs)}\right]$$

because $\mathrm{Var}\left[\widehat{\mathcal{C}}_i^{(abs)}\right] = \mathrm{Var}\left[\widehat{\mathcal{C}}_i^{(bs)}\right]$.

In general, the parameter α^* is not known exactly, so estimation is needed along with the following results.

Lemma:

$$\mathrm{Var}\left[e^{\widehat{\mathcal{S}}_i}+e^{\widehat{\mathcal{S}}_i^{(a)}}\right]=2\left(e^{\lambda\widehat{\nu}T}-2e^{2\lambda\overline{\nu}T}+e^{\lambda T(e^{a+b}-1)}\right),$$

where $\widehat{\nu}=(\exp(2b)-\exp(2a))/(2(b-a))-1$ *and* $\overline{\nu}=(\exp(b)-\exp(a))/(b-a)-1$.

Proof: Follows from properties of the antithetic pair $\left(\widehat{\mathcal{S}}_i,\widehat{\mathcal{S}}_i^{(a)}\right)$.

Lemma: *An unbiased estimator for* α^* *is*

$$\widehat{\alpha} = \frac{n}{n-1}\,\frac{\overline{XY}_n - \overline{X}_n\overline{Y}_n}{\sigma_Y^2}, \tag{8.37}$$

where $\overline{X}_n = \sum_{i=1}^{n} X_i/n$ *is the sample mean, simlarly for* $\overline{XY}_n$ *and* $\overline{Y}_n$.

Proof: Basically, the condition for an unbiased estimate $E[\widehat{\alpha}] = \alpha^*$ can be shown to be true.

Remark: *For more details on removing higher order biases, see Zhu and Hanson (2005).*

Monte Carlo Options pseudo-Algorithm with Antithetic and Control Variance (ACV) Reduction Techniques:

```
for $i=1{:}n$
    Randomly generate $P_i$ by Inverse Transform Method;
    Randomly generate IID $U_{i,j}$, $j=1{:}P_i$;
    Set $\widehat{\mathcal{S}}_i = aP_i + (b-a)\sum_{j=1}^{P_i} U_{i,j}$;
    Set $\widehat{\mathcal{S}}_i^{(a)} = (a+b)P_i - \widehat{\mathcal{S}}_i$;
    Set $\mathcal{C}_i^{(\mathrm{bs})} = \mathcal{C}^{(\mathrm{bs})}\left(S_0\exp\left(\widehat{\mathcal{S}}_i - \lambda_0\overline{\nu}T\right), T\right)$;
    Set $\mathcal{C}_i^{(abs)} = \mathcal{C}^{(\mathrm{bs})}\left(S_0\exp\left(\widehat{\mathcal{S}}_i^{(a)} - \lambda_0\overline{\nu}T\right), T\right)$;
    Set $X_i = 0.5\left(\mathcal{C}_i^{(\mathrm{bs})} + \mathcal{C}_i^{(abs)}\right)$;
    Set $Y_i = 0.5\left(\exp(\widehat{\mathcal{S}}_i) + \exp\left(\widehat{\mathcal{S}}_i^{(a)}\right)\right)$;
end %for i
Compute $\widehat{\alpha}$ according to (8.37);
Set $\widehat{Z}_n = \frac{1}{n}\sum_{i=1}^{n} X_i - \widehat{\alpha}(\frac{1}{n}\sum_{i=1}^{n} Y_i - e^{\lambda_0\overline{\nu}T})$;
Estimate bias $\widehat{\mathcal{B}}_n$ as in Zhu-Hanson (2005);
Get European call $\widehat{\mathcal{Z}}_n = \widehat{Z}_n - \widehat{\mathcal{B}}_n$;
Get European put $\widehat{\mathcal{P}}$ by Put-Call Parity.
```

Table 1: *Comparison of Option Prices by ACV Monte Carlo*

$\frac{K}{S_0}$	$\mathcal{C}$	$\mathcal{P}$	ϵ	$\mathcal{C}^{(\mathrm{bs})}$	$\mathcal{P}^{(\mathrm{bs})}$	$\mathcal{C}^*$	$\mathcal{P}^*$
0.8	269.81	0.01	2.e-3	269.80	2.e-6	269.82	0.02
0.9	132.36	1.45	0.03	130.98	0.07	132.39	1.47
1.0	40.07	20.27	0.11	30.49	10.69	40.05	20.25
1.1	5.49	76.60	0.06	1.13	72.24	5.50	76.61
1.2	0.31	147.17	0.01	4.e-3	146.87	0.32	147.19

Option parameters: $K=1000$, $r_0=0.1$, $T=0.2$. **S&P 500 estimated parameters ('88-'03):** $\sigma_0=0.1074$, $\lambda_0=64$, $a=-0.028$, $b=0.026$. Simulation count $n=10,000$. Here, $\epsilon = \sigma_{\widehat{Z}_n} = \sigma_Z/\sqrt{n}$. The $\mathcal{C}^*$ and $\mathcal{P}^*$ values are obtained by more simulations, say $n = 400,000$ sample points, as a good approximation of the true values. See *sample code*.

Theorem: Jump-Diffusion European Option Prices are Bigger than Black-Scholes Option Prices (independent of the Q-mark distribution):

$$\mathcal{C}^{(\mathrm{jd})}(S_0, T; K, r_0, \sigma_0) \geq \mathcal{C}^{(\mathrm{bs})}(S_0, T; K, r_0, \sigma_0),$$

and

$$\mathcal{P}^{(\mathrm{jd})}(S_0, T; K, r_0, \sigma_0) \geq \mathcal{P}^{(\mathrm{bs})}(S_0, T; K, r_0, \sigma_0),$$

independent of the Q*-mark distribution.*

Monte Carlo Advantages [Hanson (2007), p. 266ff]:

- Error is **theoretically independent of problem dimension**, $n_x = \mathrm{dim}[\mathcal{V}]$, $\mathcal{V}$ is the Markov simulation space of points $\vec{X}$.
- Thus, there is **no curse of dimensionality**, but it is best if $n_x \geq 5$ or so and several random samples are used, i.e.,

$$\left\{X_{i,j}^{(k)} \middle| i=1:n_x, j=1:n \text{ sample points}, k=1:K \text{ samples}\right\}.$$

- It works for **complex integrands and domains**.
- It is **not too sensitive** to a reasonable sample random number generator.

Monte Carlo Disadvantages

- There are **probabilistic error bounds, not strict errors bounds** that cannot be exceeded, e.g., 32% of samples can exceed standard error, $\sigma_f/\sqrt{n} \simeq \widehat{\sigma}_n/\sqrt{n}$.
- **Irregularity** of $F(\vec{x})$ is **not considered**, so **missed spikes or outliers** are possible.
- **Generating many large random sample sets** for high accuracy can be **costly** in computer and user time.
- **Interplay of functions and volumes can be very complex**.

{*Caution: Any advantages and disadvantages are subject to testing and performance evaluation in each case.*}

Monte Carlo Test Ratios:

When comparing two different Monte Carlo methods, one with with variance σ_1^2 and another with σ_2^2, both likely to be estimated values, then compare the methods with the **variance reduction ratio** from method 1 relative to 2,

$$\mathrm{VRR}_{1,2} = \sigma_1^2/\sigma_2^2, \tag{8.38}$$

that is, method 2 is the better variance reducer if $\mathrm{VRR}_{1,2} > 1$ and significantly larger.

Also it is necessary to check on the computational costs of the variance reduction so they are not excessive, i.e., using the **computational cost ratio**

$$\mathrm{CCR}_{1,2} = \tau_1/\tau_2, \tag{8.39}$$

where τ_1 is the computational cost (e.g., CPU time) of the first method and τ_2 for the second method.

Summary of Lecture 8?

1.

2.

3.

4.

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