Math 574 Applied Optimal Control – Hanson – Fall 2006

(Stochastic Processes and Control for Jump-Diffusions)

Homework 1 – Jump-Diffusions: Basic Properties (see Chapter 1 Text)

Homework due 04 October 2006 in class.

Justify all steps by supplying the reason(s).
See corrections, 24 February 2006.

1. Show formally that

$$\phi_{dW(t)}(w) \overset{\text{dt}}{=} \delta(w) + \frac{1}{2} dt \delta''(w) \,, \tag{1}$$

i.e., has a delta-density in the generalized sense, where $\delta(x)$ is the Dirac delta function (0.158) and $\delta''(x)$ is its 2nd derivative (0.163), by showing that

$$\mathrm{E}[f(dW(t))] = \int_{\infty}^{+\infty} \phi_{dW(t)}(w) f(w) dw \overset{\text{dt}}{=} f(0) + \frac{1}{2} dt f''(0) \,,$$

i.e., to precision-dt, neglecting terms $o(dt)$. Assume that $f(w)$ is three times continuously differentiable, with $f(w)$ and its derivatives vanishing sufficiently at infinity.
{*Hint: Only a formal expansion of $f(w)$ should be needed here, the exponential properties of $\phi_{dW(t)}(w)$ ensure sufficient uniformity to allow expansion and truncation with respect to dt inside the integral.*}

2. Show the following characteristic function (Fourier transform) formulas in the constant coefficient case, (you need only assume that the imaginary unit $i \equiv \sqrt{-1}$ is a constant with $i^2 = -1$ when integrating for the expectation or that $\zeta = i \cdot z$ can be treated the same as a real variable):

   (a) for the *Gaussian process* with time-linear drift, $G(t) = \mu_0 t + \sigma_0 W(t)$, where μ_0 and $\sigma_0 > 0$ are constants,

   $$C[G](z) \equiv \mathrm{E}[\exp(izG(t))] = \exp\left(iz\mu_0 t - z^2\sigma_0^2 t/2\right) \,;$$

   (b) for the Poisson process, $\nu_0 P$, with constant jump rate $\lambda_0 > 0$ and constant jump amplitude ν_0,

   $$C[\nu_0 P](z) \equiv \mathrm{E}[\exp(iz\nu_0 P(t))] = \exp\left(\lambda_0 t \left(\exp(iz\nu_0) - 1\right)\right) \,;$$

   (c) and finally for the jump-diffusion process $X(t) = \mu_0 t + \sigma_0 W(t) + \nu_0 P(t)$, assuming that $W(t)$ and $P(t)$ are independent processes,

   $$C[X](z) \equiv \mathrm{E}[\exp(izX(t))] = \exp\left(iz\mu_0 t - z^2\sigma_0^2 t/2 + \lambda_0 t \left(\exp(iz\nu_0) - 1\right)\right) \,.$$

3. Let $\{t_i : t_{i+1} = t_i + \Delta t_i, i = 0 : n, t_0 = 0; t_{n+1} = T\}$ be an variably-spaced partition of the time interval $[0, T]$ with $\Delta t_i > 0$. Show the following increment properties, justifying by giving a reason for every step, such as a property of the process or a property of expectations.

(a) Let $G(t) = \mu_0 t + \sigma_0 W(t)$ and $\Delta G(t_i) \equiv G(t_i + \Delta t_i) - G(t_i)$with $\mu_0 > 0$ and $\sigma_0 > 0$ constants, then show

$$\mathrm{Cov}[\Delta G(t_i), \Delta G(t_j)] = \sigma_0^2 \Delta t_i \, \delta_{i,j} \; ,$$

for $i, j = 0 : n$, where $\delta_{i,j}$ is the Kronecker delta.

(b) Let $H(t) = \nu_0 P(t)$ and $\Delta H(t_i) \equiv H(t_i + \Delta t_i) - H(t_i)$, with $\lambda_0 > 0$ and ν_0 constants, then show

$$\mathrm{Cov}[\Delta H(t_i), \Delta H(t_j)] = \nu_0^2 \lambda_0 \Delta t_i \delta_{i,j} \; ,$$

for $i, j = 0 : n$.

(c) Let $\Delta W(t_i) \equiv W(t_i + \Delta t_i) - W(t_i)$, but $\Delta^\theta W(t_i) \equiv W(t_i + \theta \Delta t_i) - W(t_i)$ with $0 < \theta < 1$, then show

$$\mathrm{Cov}[\Delta W(t_i), \Delta^\theta W(t_j)] = \theta \Delta t_i \, \delta_{i,j} \; ,$$

for $i, j = 0 : n$.

4. (a) Show that when $0 \le s \le t$ that

$$\mathrm{E}\left[W^3(t) \;\middle|\; W(r), 0 \le r \le s \right] = W^3(s) + 3(t - s)W(s) \; ,$$

justifying every step with a reason, such as a property of the process or a property of conditional expectations.

(b) Use this result to verify the martingale form

$$\mathrm{E}\left[W^3(t) - 3tW(t) \;\middle|\; W(r), 0 \le r \le s \right] = W^3(s) - 3sW(s).$$

{Remark: The general technique is to seek the expectation of mth power in the separable form,

$$E\left[M_W^{(m)}(W(t), t) \;\middle|\; W(r), 0 \le r \le s \right] = M_W^{(m)}(W(s), s) \; ,$$

where

$$M_W^{(m)}(W(t), t) = W^m(t) + \sum_{k=0}^{m-1} \alpha_k(t) W^k(t) \; ,$$

satisfied for the sequence of functions $\{\alpha_0(t), \ldots, \alpha_{m-1}(t)\}$*, that can be recursively solved using the separable form* $\alpha_k(t)$ *in the order* $k = 0 : m - 1$*; or just use the binomial theorem. Obviously,* $m = 3$ *here.}*

5. (a) Verify that when $0 \le s \le t$ and $\lambda_0 > 0$ that

$$\mathrm{E}\left[P^2(t) \;\middle|\; P(r), 0 \le r \le s \right] = P^2(s) + 2\lambda_0(t - s)P(s) + \lambda_0(t - s)(1 + \lambda_0(t - s)) \; ,$$

justifying every step with a reason, such as a property of the process or a property of conditional expectations.

(b) Find the time polynomials $\alpha_0(t)$ and $\alpha_1(t)$ such that

$$M_P^{(2)}(t) = P^2(t) + \alpha_1(t)P(t) + \alpha_0(t)$$

is a Martingale.
{Remark: The primary martingale property is that $E[X(t)|X(r), 0 \le r \le s] = X(s)$ *for some process* $X(t)$ *and in this case* $X(t) = f(P(t))$*, but there are also additional technical conditions to define a martingale form.}*