1. Show formally that
\[ \phi_{dW(t)}(w) \overset{dt}{=} \delta(w) + \frac{1}{2} dt \delta''(w), \]
i.e., has a delta-density in the generalized sense, where \( \delta(x) \) is the Dirac delta function (0.158) and \( \delta''(x) \) is its 2nd derivative (0.163), by showing that
\[ E[f(dW(t))] = \int_{-\infty}^{+\infty} \phi_{dW(t)}(w)f(w)dw \overset{dt}{=} f(0) + \frac{1}{2} dt f''(0), \]
i.e., to precision-\( dt \), neglecting terms \( o(dt) \). Assume that \( f(w) \) is three times continuously differentiable, with \( f(w) \) and its derivatives vanishing sufficiently at infinity.
{Hint: Only a formal expansion of \( f(w) \) should be needed here, the exponential properties of \( \phi_{dW(t)}(w) \) ensure sufficient uniformity to allow expansion and truncation with respect to \( dt \) inside the integral.}

2. Show the following characteristic function (Fourier transform) formulas in the constant co-efficient case, (you need only assume that the imaginary unit \( i \equiv \sqrt{-1} \) is a constant with \( i^2 = -1 \) when integrating for the expectation or that \( \zeta = i \cdot z \) can be treated the same as a real variable):
   (a) for the Gaussian process with time-linear drift, \( G(t) = \mu_0 t + \sigma_0 W(t) \), where \( \mu_0 \) and \( \sigma_0 > 0 \) are constants,
   \[ C[G](z) \equiv E[\exp(izG(t))] = \exp(iz\mu_0 t - z^2 \sigma_0^2 t/2) ; \]
   (b) for the Poisson process, \( \nu_0 P \), with constant jump rate \( \lambda_0 > 0 \) and constant jump amplitude \( \nu_0 \),
   \[ C[\nu_0 P](z) \equiv E[\exp(iz\nu_0 P(t))] = \exp(\lambda_0 t (\exp(iz\nu_0) - 1)) ; \]
   (c) and finally for the jump-diffusion process \( X(t) = \mu_0 t + \sigma_0 W(t) + \nu_0 P(t) \), assuming that \( W(t) \) and \( P(t) \) are independent processes,
   \[ C[X](z) \equiv E[\exp(izX(t))] = \exp\left(iz\mu_0 t - z^2 \sigma_0^2 t/2 + \lambda_0 t (\exp(iz\nu_0) - 1)\right) . \]

3. Let \( \{t_i : t_{i+1} = t_i + \Delta t_i, i = 0 : n, t_0 = 0; t_{n+1} = T\} \) be an variably-spaced partition of the time interval \([0, T]\) with \( \Delta t_i > 0 \). Show the following increment properties, justifying by giving a reason for every step, such as a property of the process or a property of expectations.
(a) Let \( G(t) = \mu_0 t + \sigma_0 W(t) \) and \( \Delta G(t_i) \equiv G(t_i + \Delta t_i) - G(t_i) \) with \( \mu_0 > 0 \) and \( \sigma_0 > 0 \) constants, then show
\[
\text{Cov}[\Delta G(t_i), \Delta G(t_j)] = \sigma_0^2 \Delta t_i \delta_{i,j},
\]
for \( i, j = 0 : n \), where \( \delta_{i,j} \) is the Kronecker delta.

(b) Let \( H(t) = \nu_0 P(t) \) and \( \Delta H(t_i) \equiv H(t_i + \Delta t_i) - H(t_i) \), with \( \lambda_0 > 0 \) and \( \nu_0 \) constants, then show
\[
\text{Cov}[\Delta H(t_i), \Delta H(t_j)] = \nu_0^2 \lambda_0 \Delta t_i \delta_{i,j},
\]
for \( i, j = 0 : n \).

(c) Let \( \Delta W(t_i) \equiv W(t_i + \Delta t_i) - W(t_i) \), but \( \Delta^\theta W(t_i) \equiv W(t_i + \theta \Delta t_i) - W(t_i) \) with \( 0 < \theta < 1 \), then show
\[
\text{Cov}[\Delta W(t_i), \Delta^\theta W(t_j)] = \theta \Delta t_i \delta_{i,j},
\]
for \( i, j = 0 : n \).

4. (a) Show that when \( 0 \leq s \leq t \) that
\[
\mathbb{E} \left[ W^3(t) \mid W(r), 0 \leq r \leq s \right] = W^3(s) + 3(t-s)W(s),
\]
justifying every step with a reason, such as a property of the process or a property of conditional expectations.

(b) Use this result to verify the martingale form
\[
\mathbb{E} \left[ W^3(t) - 3tW(t) \mid W(r), 0 \leq r \leq s \right] = W^3(s) - 3sW(s).
\]

Remark: The general technique is to seek the expectation of \( m \)th power in the separable form,
\[
\mathbb{E} \left[ M_W^{(m)}(W(t),t) \mid W(r), 0 \leq r \leq s \right] = M_W^{(m)}(W(s),s),
\]
where
\[
M_W^{(m)}(W(t),t) = W^m(t) + \sum_{k=0}^{m-1} \alpha_k(t)W^k(t),
\]
satisfied for the sequence of functions \( \{\alpha_0(t), \ldots, \alpha_{m-1}(t)\} \), that can be recursively solved using the separable form \( \alpha_k(t) \) in the order \( k = 0 : m-1 \); or just use the binomial theorem. Obviously, \( m = 3 \) here.

5. (a) Verify that when \( 0 \leq s \leq t \) and \( \lambda_0 > 0 \) that
\[
\mathbb{E} \left[ P^2(t) \mid P(r), 0 \leq r \leq s \right] = P^2(s) + 2\lambda_0(t-s)P(s) + \lambda_0(t-s)(1 + \lambda_0(t-s)),
\]
justifying every step with a reason, such as a property of the process or a property of conditional expectations.

(b) Find the time polynomials \( \alpha_0(t) \) and \( \alpha_1(t) \) such that
\[
M_P^{(2)}(t) = P^2(t) + \alpha_1(t)P(t) + \alpha_0(t)
\]
is a Martingale.

Remark: The primary martingale property is that \( \mathbb{E}[X(t)|X(r), 0 \leq r \leq s] = X(s) \) for some process \( X(t) \) and in this case \( X(t) = f(P(t)) \), but there are also additional technical conditions to define a martingale form.