Math 574 Applied Optimal Control – Hanson – Fall 2006

(Stochastic Processes and Control for Jump-Diffusions)

Homework 2 – Stochastic Diffusion Integration
(See Chapter 2 of Text; See also Chapter 0 for Preliminaries)

Homework due 11 October 2006 in class.

1. Justify the general form, by Itô mean square convergence:

$$(dt)^p(dW)^q(t) \underset{sym}{\overset{ims}{=}} \delta_{p,0}\delta_{q,0} + dW(t)\delta_{p,0}\delta_{q,1} + dt(\delta_{p,1}\delta_{q,0} + \delta_{p,0}\delta_{q,2}) \ ,$$

when p and q are non-negative integers. {Remark: It may be assumed that the cases $2p + q = 0 : 2$ are well-known, so need to show mean square convergence results for $2p + q \leq 3$.

2. Show the limit in the mean square for

$$I\left[(dt)^\alpha\right](t) \equiv \int_0^t (dt)^\alpha dW(s) \overset{ims}{=} 0 \ ,$$

provided $\alpha > 0$ and is real (i.e., not necessarily an integer). {*Hint: See Lemma 2.18 for the case* $\alpha = 1$.}

3. Formally show that the θ-rule leads to

$$E\left[\int_0^t g(W(s))dW(s)\right] \overset{\theta-\text{ms}}{=} I^{(\theta)}[g(W)](t) = \theta \int_0^t E[g'(W(s))]ds,$$

where $0 \leq \theta \leq 1$, assuming the basic θ-rule approximation for the stochastic integral is

$$\int_0^t g(W(s))dW(s) \simeq I_n^{(\theta)}[g(W)](t) \equiv \sum_{i=0}^{n} g_{i+\theta}\Delta W_i,$$

where g has a bounded mean square expectation (see the text), $g_{i+\theta} = g(W_{i+\theta}) = g(W(t_{i+\theta}))$ (see the text), and assuming that g satisfies the two-term Taylor approximation

$$g(w_0 + \Delta W) = g(w_0) + g'(w_0)\Delta W + \mathrm{O}^2(\Delta W),$$

sufficiently uniform with respect to the density $\phi_{DW(t)}(w)$ on $(-\infty, +\infty)$ to allow termwise expectations, provided you can show that $\mathrm{E}[(\Delta^\theta W_i)^m] = \mathrm{O}^2(\theta\Delta t_i)$ for sufficiently small Δt_i. See also the θ-decomposition in the text of DW_i. {*Remark: Thus, this demonstrates that the Itô sense Theorem 2.17 is generally limited to* $\theta = 0$.}

4. **Computationally confirm** the mean square limit for Itô's most fundamental stochastic integral:

$$\int_0^t (dW)^2(s) \overset{ims}{=} t \ ,$$

by demonstrating that the Itô forward integration approximating sum

$$I_n[dW](t) = \sum_{i=0}^{n} (\Delta W_i)^2$$

gives a close approximation to t for sufficiently large n. Apply a modification of the algorithm of the Wiener Program A.7 in Appendix A generating Figure 1.1 to the approximation $I_n[dW](t)$. Use $n = 1000$ and $n = 10000$ sample step sizes, plotting the $I_n[dW](t)$ with the limit t versus t for $t \in [0, 2]$. Plot separately the errors for each n between the approximation sum and the exact IMS answer. Also report the standard deviation (`std` in MATLAB) of the errors for each n. Characterize the convergence on the average by assuming that the standard deviation satisfies the simple rule $std_m \simeq C/n^\beta$ as $n \to \infty$, where $m \equiv \ln(n)$ and find the average convergence rate β from the two sample step sizes n.

5. **Computationally confirm** the mean square limit for Itô's other very fundamental stochastic integral:

$$\int_0^t W(s)dW(s) \overset{ims}{=} I^{(\mathrm{ims})}[W](t) = \frac{1}{2}\left(W^2(t) - t\right)$$

by demonstrating that the Itô forward integration approximating sum

$$I_n[W](t) = \sum_{i=0}^{n} W_i \Delta W_i$$

gives a close approximation to $(W^2(t) - t)/2$ for sufficiently large n. Apply a modification of the algorithm of Program A.7 in Appendix A used in generating Figure 1.1, to the approximation $I_n[W](t)$. Use Use $n = 1000$ and $n = 10000$ sample sizes, plotting the approximation $I_n[W](t)$ and the error $E_n[W](t) = I_n[W](t) - (W^2(t) - t)/2$ versus t for $t \in [0, 2]$. Plot separately the errors for each n between the approximation sum and the exact IMS answer. Also report the standard deviation (`std` in MATLAB) of the errors for each n. As in the prior exercise, is this average rate a sublinear convergence rate, i.e., $0 < \beta < 1$ {*Remark: $\beta = 1$ is a linear rate*}.

6. **Computationally confirm** the mean square limit for Itô's another more obvious fundamental stochastic integral:

$$\int_0^t ds dW(s) \overset{ims}{=} I^{(\mathrm{ims})}[dt](t) = 0$$

by demonstrating that the Itô forward integration approximating sum

$$I_n[dt](t) = \sum_{i=0}^{n} \Delta t_i \Delta W_i$$

gives a close approximation to 0 for sufficiently large n. Apply a modification of the algorithm of Program A.7 in Appendix A, used in generating Figure 1.1, to the approximation $I_n[dt](t)$. Use $n = 1000$ and $n = 10000$ sample sizes, plotting the common value of the approximation and error $I_n[dt](t) = E_n[dt](t)$ and the noise $W(t)$ for $t \in [0, 2]$. Plot separately the errors for each n between the approximation sum and the exact IMS answer. Also report the standard deviation (`std` in MATLAB) of the errors for each n. Does the larger value of n make Itô's stochastic integration model more convincing than the smaller value?