Math 574 Applied Optimal Control – Hanson – Fall 2006
(Stochastic Processes and Control for Jump-Diffusions)

Homework 3 – Jump-Diffusion SDEs
(Chapter 4 Text; See also Chapter 0 for Preliminaries)

Homework due 8 November 2006 in class.

1. Derive the Itô stochastic integral formulas for

$$\int_0^t \cos(aW(s))dW(s) \quad \text{and} \quad \int_0^t \sin(aW(s))dW(s)\ , \tag{1}$$

where a is a real constant $\neq 0$.

2. Solve the following (Itô) diffusion SDE for $X(t)$, $\mathrm{E}[X(t)]$, and $\mathrm{Var}[X(t)]$,

$$dX(t) = \left(a\sqrt{X(t)} + b^2/4\right)dt + b\sqrt{X(t)}dW(t)\ , \tag{2}$$

where a and b are real constants, and $X(0) = x_0 > 0$,with probability one.
{*Hint: seek a transformation* $Y(t) = F(X(t))$ *for some* F *such that* $Y(t)$ *satisfies a constant coefficient SDE.*}

3. Find $X(t) = F(P(t))$ if

$$\int_0^t X(s)dP(s) = e^{cP(t)}\ln(aP(t) + b) - \ln(b)\ , \tag{3}$$

where a, b and c are real constants.

4. Solve the following jump SDE for $X(t)$, $\mathrm{E}[X(t)]$, and $\mathrm{Var}[X(t)]$,

$$dX(t) = aX^2(t)dt - \frac{bX^2(t)}{1 + bX(t)}dP(t)\ , \tag{4}$$

where $a < 0$, $b > 0$ and $c > 0$ are constants such that $\mathrm{E}[P(t)] = ct$, while $X(0) = x_0 > 0$, with probability one.
{*Hint: seek a transformation* $Y(t) = F(X(t))$ *for some* F *such that* $Y(t)$ *satisfies a constant coefficient SDE. The answer may be left as a Poisson distribution sum.*}

5. Show that the (Itô) jump-diffusion SDE for $X(t)$,

$$dX(t) = f(X(t))dt + bX^a(t)dW(t) + h(X(t))dP(t)\ , \tag{5}$$

can be transformed by $Y(t) = F(X(t))$ to a **constant coefficient SDE**, where b and $a \neq 1$ are real constants, and $X(0) = x_0 > 0$, with probability one. In a proper answer, the power forms of $f(X(t))$ and $h(X(t))$ must be derived from the constant coefficient SDE condition. Also, what is the answer when $a = 1$?

6. A martingale in continuous time satisfies the essential property that

$$\mathrm{E}[M(t)|M(s)] = M(s),$$

for all $0 \leq s < t$ provided its absolute value has finite expectation, i.e., $\mathrm{E}[|M(t)|] < \infty$ for all $t \geq 0$, plus some other technical properties (see Mikosch (1998), for instance).

   (a) Show that

   $$M_1(t) = \ln(X(t)) - \mathrm{E}[\ln(X(t))]$$

   is a martingale where $Y(t) = \ln(X(t))$ symbolically satisfies the solution to the general linear jump-diffusion SDE transformed to state-independent SDE form (like (4.76)).

   (b) Derive the function $\beta(t)$ that makes

   $$M_2 = \beta(t)X(t)$$

   a martingale if $X(t)$ symbolically satisfies the linear jump-diffusion SDE.