Homework 3 – Jump-Diffusion SDEs

(Chapter 4 Text; See also Chapter 0 for Preliminaries)

Homework due 8 November 2006 in class.

1. Derive the Itô stochastic integral formulas for
\[
\int_0^t \cos(aW(s))dW(s) \quad \text{and} \quad \int_0^t \sin(aW(s))dW(s),
\]
where \(a\) is a real constant \(\neq 0\).

2. Solve the following (Itô) diffusion SDE for \(X(t)\), \(E[X(t)]\), and \(\text{Var}[X(t)]\),
\[
dX(t) = \left(a\sqrt{X(t)} + b^2/4\right)dt + b\sqrt{X(t)}dW(t),
\]
where \(a\) and \(b\) are real constants, and \(X(0) = x_0 > 0\), with probability one.

Hint: seek a transformation \(Y(t) = F(X(t))\) for some \(F\) such that \(Y(t)\) satisfies a constant coefficient SDE.

3. Find \(X(t) = F(P(t))\) if
\[
\int_0^t X(s)dP(s) = e^{\epsilon P(t)} \ln(aP(t) + b) - \ln(b),
\]
where \(a\), \(b\) and \(c\) are real constants.

4. Solve the following jump SDE for \(X(t)\), \(E[X(t)]\), and \(\text{Var}[X(t)]\),
\[
dX(t) = aX^2(t)dt - \frac{bX^2(t)}{1 + bX(t)}dP(t),
\]
where \(a < 0\), \(b > 0\) and \(c > 0\) are constants such that \(E[P(t)] = ct\), while \(X(0) = x_0 > 0\), with probability one.

Hint: seek a transformation \(Y(t) = F(X(t))\) for some \(F\) such that \(Y(t)\) satisfies a constant coefficient SDE. The answer may be left as a Poisson distribution sum.

5. Show that the (Itô) jump-diffusion SDE for \(X(t)\),
\[
dX(t) = f(X(t))dt + bX^a(t)dW(t) + h(X(t))dP(t),
\]
can be transformed by \(Y(t) = F(X(t))\) to a constant coefficient SDE, where \(b\) and \(a \neq 1\) are real constants, and \(X(0) = x_0 > 0\), with probability one. In a proper answer, the power forms of \(f(X(t))\) and \(h(X(t))\) must be derived from the constant coefficient SDE condition. Also, what is the answer when \(a = 1\)?

6. A martingale in continuous time satisfies the essential property that
\[
E[M(t)|M(s)] = M(s),
\]
for all \(0 \leq s < t\) provided its absolute value has finite expectation, i.e., \(E[|M(t)|] < \infty\) for all \(t \geq 0\), plus some other technical properties (see Mikosch (1998), for instance).

(a) Show that
\[
M_1(t) = \ln(X(t)) - E[\ln(X(t))]
\]
is a martingale where \(Y(t) = \ln(X(t))\) symbolically satisfies the solution to the general linear jump-diffusion SDE transformed to state-independent SDE form (like (4.76)).

(b) Derive the function \(\beta(t)\) that makes
\[
M_2 = \beta(t)X(t)
\]
a martingale if \(X(t)\) symbolically satisfies the linear jump-diffusion SDE.