1. Consider another simple lumped model of a leaky reservoir given by
\[
\dot{X}(t) = -aX(t) + U(t), \quad X(0) = x_0,
\]
where \(X(t)\) is the depth of the reservoir, \(U(t)\) is the net flow of water per unit time into the reservoir at time \(t\) constrained and \(a > 0\) is the rate of leakage and usage. The net inflow is constrained pointwise \(0 \leq U(t) \leq M\) for all \(0 < t \leq t_f\) and also cumulatively by
\[
\int_0^{t_f} U(t)\,dt = K > 0,
\]
where \(K, M\) and \(t_f\) are fixed constants, such that \(K \leq M \cdot t_f\) for consistency. Find the optimal control law that maximizes only the final depth,
\[
J[X] = bX(t_f)
\]
with \(b > 0\) and the optimal state \(X^*(t)\).

2. For the deterministic linear first order dynamics,
\[
\dot{X}(t) = -\mu_0 X(t) + \beta_0 U(t), \quad t > 0, \quad \text{given } X(0) = x_0 \neq 0, \quad \mu_0 > 0, \quad \beta_0 \neq 0,
\]
and quadratic performance measure,
\[
V[U] = \frac{r_0}{2} \int_0^{t_f} U^2(t)\,dt, \quad r_0 > 0,
\]
find the optimal state trajectory and optimal (unconstrained) control to bring the state from the initial state to the final state \(x_f\) in \(t_f\) seconds while minimizing the functional \(V[U]\) with respect to the control \(u\), with the answer depending on the parameter set \(\{x_0, x_f, t_f, \mu_0, \beta_0, r_0\}\). Note that the final state and time are fixed.