Math 574 Applied Optimal Control – Hanson – Fall 2006

(Stochastic Processes and Control for Jump-Diffusions)

Homework 7 – Stochastic Control (Chapters 7 of Text; See also Chapter 0 for Preliminaries)

Homework due 08 December 2005 in class.
Acknowledge consultation with others, even if none, else receive a grade discount.

1. For the linear jump-diffusion dynamics

$$dX(t) = (\mu_0 X(t) + \beta_0 U(t))dt + \sigma_0 dW(t) + \nu_0 X(t)dP(t),$$

for $0 \le t \le t_f$ and state $X(0) = x_0 > 0$ and the control process $-\infty < U(t) < +\infty$ is unconstrained. The coefficients $\mu_0 \neq 0$, $\beta_0 \neq 0$, $\sigma_0 > 0$, $\nu_0 \neq 0$ and $\lambda_0 > 0$ are constants, where $\mathrm{E}[dP(t)] = \lambda_0 dt$. The costs are quadratic, i.e.,

$$V[X,U](X(t),t) = \frac{1}{2}\int_t^{t_f} \left(q_0 X^2(s) + r_0 U^2(s)\right) ds + \frac{1}{2} S_f X^2(t_f)$$

for $q_0 > 0$, $r_0 > 0$, and $S_f > 0$,

   (a) Derive the PDE of Stochastic Dynamic Programming for the optimal expected value:
$$v^*(x,t) = \min_u \left[\mathrm{E}\left[V[X,U](X(t),t)\,|X(t) = x, U(t) = u\right]\right],$$
using the Principle of Optimality;

   (b) Specify the final condition for $v^*(x,t)$;

   (c) Formally find the optimal (unconstrained) control $u^*(x,t)$ in terms of the *shadow "cost"* $v^*_x(x,t)$;

2. Derive the modifications necessary in the set of Riccati-like equations for the Linear-Quadratic Jump-Diffusion (LQJD, LQGP or JLQG) problem when the dynamics are scalar and linear (affine), i.e.,

$$dX(t) = f(X(t),U(t),t)dt + g(X(t),t)dW(t) + h(X(t),t)dP(t)\ ,$$

where

$$\mathrm{E}[dP(t)] = \lambda(t)dt\ ,$$
$$f(x,u,t) = f_{0,0}(t) + f_{1,1}(t)x + f_{1,2}(t)u\ ,$$
$$g(x,t) = g_{0,0}(t) + g_{1,1}(t)x\ ,$$
$$h(x,t) = h_{0,0}(t) + h_{1,1}(t)x\ ,$$

the jump amplitude being independent of any mark process. The running and terminal costs for a maximum objective are quadratic,

$$\begin{aligned} C(x,u,t) &= C_{0,0}(t) + C_{1,1}(t)x + C_{1,2}(t)u + 0.5C_{2,1,1}(t)x^2 + C_{2,1,2}(t)xu \\ &\quad + 0.5C_{2,2,2}(t)u^2\ , \end{aligned}$$

where $C_{02}(t) < 0$, and

$$S(x,t) = S_0(t) + S_1(t)x + 0.5 * S_2(t)x^2 \,,$$

where $S_2(t) < 0$.

If the objective is to maximize the expected total utility in the unconstrained control case, then find the coefficient functions $v_0(t)$, $v_1(t)$, $v_2(t)$, $u_0(t)$ and $u_1(t)$ in the solutions

$$v^*(x,t) = v_0(t) + v_1(t)x + 0.5v_2(t)x^2$$

and

$$u^*(x,t) = u_0(t) + u_1(t)x$$

explicitly in terms of the dynamical and cost coefficient functions. Do not try to solve the Riccati equation system for $\{v_0(t), v_1(t), v_2(t)\}$.

3. Derive the Hamilton-Jacobi-Bellman PDE for the optimal stochastic control problem (a simplified jump-diffusion optimal portfolio problem), with stochastic dynamical system,

$$dX(t) = X(t)\left(\mu_0(t) + \mu_1(t)U_1(t))dt + \sigma(t)dW(t) + \left(e^Q - 1\right)dP(t)\right) - U_2(t)dt \,,$$

where $t \in [0, t_f]$, $X(0) = x_0$, $\mathrm{E}[dP(t)] = \lambda(t)dt$, $\mathrm{Var}[W(t)] = \mathrm{E}[W^2(t)] = dt$, Q is an IID uniformly distributed mark on $[a, b]$, $a < 0 < b$, $\{\mu_0(t), \mu_1(t), \sigma(t), \lambda(t)\}$ are specified time-dependent coefficients, $X(t) \geq 0$ is the state, $\{U_1(t), U_2(t)\}$ is the control set, $0 \leq U_2(t) \leq K_2X(t)$, $K_2 > 0$, and the optimal objective is

$$v^*(x,t) = \max_{\{u_1, u_2\}}\left[\mathrm{E}\left[\int_t^{t_f} e^{-\beta(s-t)}\frac{U_2^{\gamma}(s)}{\gamma}ds + e^{-\beta(t_f-t)}\frac{X^{\gamma}(t_f)}{\gamma}\middle|\,\mathcal{C}\right]\right],$$

where $\mathcal{C} \equiv \{X(t) = x, U_1(t) = u_1(t), U_2(t) = u_2(t)\}$ is the conditioning set, $\beta > 0$ is a constant discount factor, $\gamma \in (0,1)$ is a constant utility power and the zero-state absorbing boundary condition is $v^*(0^+, t) = 0$. Determine the modified HJBE when the discount factor is time-dependent, i.e., the instantaneous discount factor $\widehat{\beta}(t)$, replacing $\beta(s-t)$ by the cumulative discount $B(t,s) \equiv \int_t^s \widehat{\beta}(r)dr$, etc.