

(Stochastic Processes and Control for Jump-Diffusions)

Homework 7 – Stochastic Control (Chapters 7 of Text; See also Chapter 0 for Preliminaries)

*Homework due 08 December 2005 in class.*

*Acknowledge consultation with others, even if none, else receive a grade discount.*

1. For the linear jump-diffusion dynamics

$$dX(t) = (\mu_0 X(t) + \beta_0 U(t))dt + \sigma_0 dW(t) + \nu_0 X(t)dP(t),$$

for  $0 \leq t \leq t_f$  and state  $X(0) = x_0 > 0$  and the control process  $-\infty < U(t) < +\infty$  is unconstrained. The coefficients  $\mu_0 \neq 0$ ,  $\beta_0 \neq 0$ ,  $\sigma_0 > 0$ ,  $\nu_0 \neq 0$  and  $\lambda_0 > 0$  are constants, where  $E[dP(t)] = \lambda_0 dt$ . The costs are quadratic, i.e.,

$$V[X, U](X(t), t) = \frac{1}{2} \int_t^{t_f} (q_0 X^2(s) + r_0 U^2(s)) ds + \frac{1}{2} S_f X^2(t_f)$$

for  $q_0 > 0$ ,  $r_0 > 0$ , and  $S_f > 0$ ,

- (a) Derive the PDE of Stochastic Dynamic Programming for the optimal expected value:

$$v^*(x, t) = \min_u [E[V[X, U](X(t), t) | X(t) = x, U(t) = u]],$$

using the Principle of Optimality;

- (b) Specify the final condition for  $v^*(x, t)$ ;

- (c) Formally find the optimal (unconstrained) control  $u^*(x, t)$  in terms of the *shadow “cost”*  $v_x^*(x, t)$ ;

2. Derive the modifications necessary in the set of Riccati-like equations for the Linear-Quadratic Jump-Diffusion (LQJD, LQGP or JLQG) problem when the dynamics are scalar and linear (affine), i.e.,

$$dX(t) = f(X(t), U(t), t)dt + g(X(t), t)dW(t) + h(X(t), t)dP(t),$$

where

$$E[dP(t)] = \lambda(t)dt,$$

$$f(x, u, t) = f_{0,0}(t) + f_{1,1}(t)x + f_{1,2}(t)u,$$

$$g(x, t) = g_{0,0}(t) + g_{1,1}(t)x,$$

$$h(x, t) = h_{0,0}(t) + h_{1,1}(t)x,$$

the jump amplitude being independent of any mark process. The running and terminal costs for a maximum objective are quadratic,

$$C(x, u, t) = C_{0,0}(t) + C_{1,1}(t)x + C_{1,2}(t)u + 0.5C_{2,1,1}(t)x^2 + C_{2,1,2}(t)xu + 0.5C_{2,2,2}(t)u^2,$$

where  $C_{02}(t) < 0$ , and

$$S(x, t) = S_0(t) + S_1(t)x + 0.5 * S_2(t)x^2,$$

where  $S_2(t) < 0$ .

If the objective is to maximize the expected total utility in the unconstrained control case, then find the coefficient functions  $v_0(t)$ ,  $v_1(t)$ ,  $v_2(t)$ ,  $u_0(t)$  and  $u_1(t)$  in the solutions

$$v^*(x, t) = v_0(t) + v_1(t)x + 0.5v_2(t)x^2$$

and

$$u^*(x, t) = u_0(t) + u_1(t)x$$

explicitly in terms of the dynamical and cost coefficient functions. Do not try to solve the Riccati equation system for  $\{v_0(t), v_1(t), v_2(t)\}$ .

3. Derive the Hamilton-Jacobi-Bellman PDE for the optimal stochastic control problem (a simplified jump-diffusion optimal portfolio problem), with stochastic dynamical system,

$$dX(t) = X(t) (\mu_0(t) + \mu_1(t)U_1(t))dt + \sigma(t)dW(t) + (e^Q - 1) dP(t) - U_2(t)dt,$$

where  $t \in [0, t_f]$ ,  $X(0) = x_0$ ,  $E[dP(t)] = \lambda(t)dt$ ,  $\text{Var}[W(t)] = E[W^2(t)] = dt$ ,  $Q$  is an IID uniformly distributed mark on  $[a, b]$ ,  $a < 0 < b$ ,  $\{\mu_0(t), \mu_1(t), \sigma(t), \lambda(t)\}$  are specified time-dependent coefficients,  $X(t) \geq 0$  is the state,  $\{U_1(t), U_2(t)\}$  is the control set,  $0 \leq U_2(t) \leq K_2X(t)$ ,  $K_2 > 0$ , and the optimal objective is

$$v^*(x, t) = \max_{\{u_1, u_2\}} \left[ E \left[ \int_t^{t_f} e^{-\beta(s-t)} \frac{U_2^\gamma(s)}{\gamma} ds + e^{-\beta(t_f-t)} \frac{X^\gamma(t_f)}{\gamma} \middle| \mathcal{C} \right] \right],$$

where  $\mathcal{C} \equiv \{X(t) = x, U_1(t) = u_1(t), U_2(t) = u_2(t)\}$  is the conditioning set,  $\beta > 0$  is a constant discount factor,  $\gamma \in (0, 1)$  is a constant utility power and the zero-state absorbing boundary condition is  $v^*(0^+, t) = 0$ . Determine the modified HJBE when the discount factor is time-dependent, i.e., the instantaneous discount factor  $\hat{\beta}(t)$ , replacing  $\beta(s-t)$  by the cumulative discount  $B(t, s) \equiv \int_t^s \hat{\beta}(r)dr$ , etc.