Math 586 Computational Finance – Hanson – Spring 2008

Homework 4 – Jump-Diffusions: SDEs, Options, Risk-Neutrality, Stochastic-Volatility and Simulations

- ***Homework now due in class 02 May 2008, OK without penalty.***
- *For computations, round final results to 4 significant digits (e.g., 12.34 or 0.01234 or 1.234$\cdot 10^4$), except round monetary values to the nearest hundredths of a dollar; use MATLAB built-in double-precision for intermediate calculations; hand in documented code with documented output.*
- *This is individual homework: you may discuss generally with others if cited, but final submission must be your own work. Also, cite any outside sources or references that you use. This includes outside code sources, which also have to be verified for correctness and accuracy.*
- *MATLAB computational solutions are recommended; Maple or Mathematica symbolic proofs or solutions are acceptable, if appropriate; you can ask Prof. Hanson for MATLAB help.*
- Mathematical Science Prelim Considerations: If you are taking any the Computational Finance questions (4 total) on this prelim, you can use the grades on the questions taken as a substitute for part or all of the questions here, depending on your performance, otherwise pro-rated according to the number of answered computational finance questions you submit, discounted from 20 points per prelim question by 25%, so that the maximum of 80 points on the prelim would be the same as the maximum of 60 points for the Math 586 questions here, provided that the prelim is given this semester.

See correction(s) in red, 29 April 2008.

1. **Simple and Compound Poisson Increment Process Central Moments:** Calculate the expectations for $\lambda = \lambda_0$ constant:

   (a) $\mathrm{E}\left[(\Delta P(t) - \lambda_0 \Delta t)^3\right]$;

   (b) $\mathrm{E}\left[(\Delta P(t) - \lambda_0 \Delta t)^4\right]$;

   (c) $\mathrm{E}\left[(\nu(t,Q)\Delta P(t;Q) - \overline{\nu}(t)\lambda_0 \Delta t)^2\right]$;

   (d) $\mathrm{E}\left[(\nu(t,Q)\Delta P(t;Q) - \overline{\nu}(t)\lambda_0 \Delta t)^3\right]$;

   putting the jump-amplitude moments in the non-central form
   $$\overline{\nu^k}(t) \equiv \mathrm{E}_Q\left[\nu^k(t,Q)\right] \text{ for } k = 1:3.$$
   {*Comment: using zero mean forms for at least the ΔP's may facilitate calculations.*}

2. **Stochastic Chain Rule, Simple Poisson Jump Change Calculations:** Show:

   (a) If $X(t) = K \cdot P^2(t)$, then $dX(t) = K \cdot (2 \cdot P(t) + 1) dP(t)$;

   (b) If $X(t) = K \cdot P^3(t)$, then $dX(t) = K \cdot (3 \cdot P^2(t) + 3 \cdot P(t) + 1) dP(t)$;

   (c) If $X(t) = K/(P(t) - 1/2)$, then $dX(t) = -K \cdot dP(t) / \left(P^2(t) - 1/4\right)$;

   where K is a constant; **you must justify all steps**.
   {*Comment: essentially, you have to find the jump in $X(t)$.*}

3. **Expectations of Partial Sums of IID Jump Marks in Compound Jump-Diffusions:** Show if $\mathcal{S}_k \equiv \sum_{i=1}^{k} Q_i$
   and the Q_i IID, then:

   (a) $\mathrm{E}_Q[\mathcal{S}_k] = k \cdot \mu_Q$, where $\mu_Q \equiv \mathrm{E}_Q[Q_i]$ for any i;

(b) $\mathrm{Var}_Q[\mathcal{S}_k] = k \cdot \sigma_Q^2$, where $\sigma_Q^2 \equiv \mathrm{Var}_Q[Q_i]$ for any i;

(c) $\mathrm{E}_{\boldsymbol{Q}}\left[(\boldsymbol{\mathcal{S}_k} - \mathrm{E}_{\boldsymbol{Q}}[\boldsymbol{\mathcal{S}_k}])^{\mathbf{3}}\right] = \boldsymbol{k} \cdot \boldsymbol{\mu_Q^{(3)}}$, where $\mu_Q^{(3)} \equiv \mathrm{E}_Q\left[(Q_i - \mu_Q)^3\right]$;

(d) What your guess for $\mathrm{E}_{\boldsymbol{Q}}\left[(\boldsymbol{\mathcal{S}_k} - \mathrm{E}_{\boldsymbol{Q}}[\boldsymbol{\mathcal{S}_k}])^{\mathbf{4}}\right]$, i.e., will it be the similar to the 3rd central moment or not, and why (beyond the fact that the power is even)?

(e) $\mathrm{E}_Q[\exp(\mathcal{S}_k)] = \mu_{\exp(Q)}^k$, where $\mu_{\exp(Q)} \equiv \mathrm{E}_Q[\exp(Q_i)]$ for any i (note that this form is used for the exponential form of the jump-diffusion solution);

{*Comment: note* $\sum_{i=1}^k C = k \cdot C$, *where* C *is a constant;* $(\sum_{i=1}^k a_i)^2 = \sum_{i=1}^k a_i \cdot \sum_{j=1}^k a_j$ *and similarly for* $(\sum_{i=1}^k a_i)^3$.}

4. **Simulations of Sample Paths for Distributed Jump-Diffusions:** Compute and plot four sample paths together for

$$S(t + \Delta t) = S(t) \cdot \exp((\mu(t) - \sigma^2(t)/2) \cdot \Delta t + \sigma(t) \cdot \Delta W(t) + \ln(1 + \nu(t, Q)) \cdot \Delta P(t; Q)),$$

with $t = t_i$ for $i = 1 : N + 1$, $t_1 = 0$, $t_{N+1} = T$ and $S(0) = S_0$.

Let $S_0 = \$100$, $\mu(t) = 0.25 \cdot (1 - 0.56 \cdot t)$, $\sigma(t) = 0.18 \cdot (1 + 0.10 \cdot t)$, $\nu(t, Q) = \exp(Q) - 1$ where Q is uniformly IID on $[a(t), b(t)] = [-0.030 \cdot (1 - 0.12 \cdot t), +0.025 \cdot (1 - 0.15 \cdot t)]$, $\boldsymbol{\lambda(t) = 60 \cdot (1 - 0.12 * t)}$, $N = 10{,}000$, and $T = 3.0$ years. You may modify Hanson's (2007) Linear-Mark-Jump-Diffusion Simulator,

`http://www.math.uic.edu/~hanson/pub/SIAMbook/MATLABCodes/linmarkjumpdiff06fig1.m`

for this problem's coefficients and parameters.

Describe, in words, your results as potential financial stock price trajectories compared to linear pure diffusion with drift (Gaussian) processes.

5. **Computation Using Merton's (1976) Jump-Diffusion Model for European Options:** Compute the European call and put option prices for the following model:

$$\boldsymbol{dS(t) = S(t) \cdot ((\mu - \overline{\nu}\lambda) \cdot dt + \sigma \cdot dW(t) + \nu \cdot dP(t; \nu))},$$

where here ν is an IID random variable (i.e., the jump-amplitude is its own mark, which in Merton's 1976 model is log-normally distributed) and the total jump-term symbolic notation means $S(t) \cdot \nu \cdot P(t; \nu) \equiv \sum_{i=1}^{P(t;\nu)} S(T_i^-) \cdot \nu_i$, where T_i is the ith jump-time with jump-amplitude ν_i, $\boldsymbol{\mu = \mu_0 + \overline{\nu}\lambda}$ here is the total jump-diffusion mean rate coefficient, σ is just the diffusion volatility coefficient, and λ is the number of jumps per year. All coefficients are constant. Please use Kevin Cheng's Global Derivatives Merton Jump-Diffusion MATLAB function code:

`http://www.global-derivatives.com/code/matlab/MertonJumpEuro.m` ,

which calls his Basic Black-Scholes European option MATLAB function multiple times:

`http://www.global-derivatives.com/code/matlab/BlackScholesEuro.m` ,

which in turn call the MATLAB Statistics Toolbox standard normal probability distribution function `normcdf` (if you do not have this toolbox, then you can write a simple (sub)function for it from the formula in MATLAB Help `erfc`:

$$\mathrm{normcdf}(x) = 0.5\mathrm{erfc}(-x/\mathrm{sqrt}(2));$$

which is too simple to need the toolbox anyway).

It is recommended, to avoid path problems, that you merge these functions into one MATLAB m-file, including `normcdf` if needed, with a beginning driver function for handling input and output.

(a) Compute results for both call and put;

(b) Do this with the two values of `Vol` given below, meaning a total of 4 cases;

(c) Compare results with respect to `Vol`;

(d) Use the following input:

$S_0 = \$95$ is the initial asset price,

$K = \$100$ is the strike price,

$r = 0.05$ (5%) is the risk free interest rate,

$T = 1.0$ years is the time to maturity,

$\texttt{Vol} = \{0.25 \text{ and } 0.30\}$, or $\{(25\%) \text{ and } (30\%)\}$, respectively, are two possible total jump-diffusion volatilities of the asset $\left(\texttt{Vol}^2 \equiv \sigma_d^2 + \lambda_j(\mu_j^2 + \sigma_j^2)\right)$, where j subscripts denote jump and d denotes diffusion),

$\texttt{Gamma} = 0.60$ (60%) is percent of jump-diffusion volatility explained by jumps $\left(\texttt{Gamma} \equiv \sqrt{1 - \sigma_d^2/\texttt{Vol}^2}\right)$,

$\texttt{Jumps} = 100$ is the number of jumps per year (λ_j),

$\texttt{MaxIter} = \mathbf{100}$ is the number of total jumps included (note the Poisson probabilities will be very small for a larger jump numbers).

{*Comments: Note that Merton assumed a log-normal jump-amplitude distribution and Cheng along with Merton used approximate average implied volatilities to avoid proper simulations. The short problem description at*

http://www.global-derivatives.com/index.php?option=com_content&task=view&id=15&Itemid=31 , item 5), has errors and is inconsistent with Cheng's code and Merton's 1976 paper. See item 2) for the Black-Scholes model.}

6. **Stochastic Volatility Simulations:** There are several kinds of stochastic volatility models, but a very popular one used with option models is due to Heston (1993) and originally came from interest rate models of Cox, Ingersoll and Ross (1985), called the CIR model. The Heston stochastic volatility model uses a so-called square root noise for the stock-price variance $V(t)$ so $\sigma_s(t) = \sqrt{V(t)}$ with a similar term in the variance SDE and a linear (affine), mean-reverting term for the drift of the variance to level θ at rate κ. The SDE of the stock-price variance or equation of stochastic volatility is

$$dV(t) = \kappa \cdot (\theta - V(t)) \cdot dt + \sigma_v \cdot \sqrt{V(t)} \cdot dW_v(t), \quad V(0) = V_0 > 0, \qquad (SV)$$

where σ_v is called the *volatility of the volatility*, or the *vol of vol* for short. The full stochastic-volatility, jump-diffusion (SVJD) model for the option pricing problem is given in Yan-Hanson (2006, 2007) and in Hanson (2008) for the SVJD optimal portfolio and consumption problem. This question only concerns the simulating solutions of stochastic volatility equation above.)

Equation (SV), unlike the linear jump-diffusion SDE with removable stock price ($S(t)$, cannot be so simplified with a transformation as simple as the logarithm, if at all. Thus, there is a real question as to whether the general stochastic chain rule of Itô through the application of a Taylor's expansion is legitimate since the square root is not differentiable as $V(t) \to 0^+$. Hence, the following modified difference equation should be used:

$$V(t + \Delta t) = \max\left(V(t) + \kappa \cdot (\theta - V(t)) \cdot \Delta t + \sigma_v \cdot \sqrt{V(t)} \cdot \Delta W_v(t) \ , \ \epsilon_v\right), \qquad (\Delta SV)$$

where here $\boldsymbol{\epsilon_v = 0.5 \cdot \sqrt{\Delta t}}$ is taken so $\Delta t/\epsilon_v \to 0^+$ as $\Delta t \to 0^+$ and is required for validity of the Taylor approximations.

Let $V_0 = 0.10$ **per year**, $\kappa = 1.0$ per year, $\theta = 0.01$ **per year**, $\sigma_v = 0.20$ per year, $T = 2$ years and $N = 10000 \cdot T$.

(a) Simulate the solution to (ΔSV) with the given parameters.

(b) Simulate the deterministic ($\sigma_v = 0$) solution:

$$\text{Vdet}(t) = V_0 \cdot \exp(-\kappa \cdot t) + \theta \cdot (1 - \exp(-\kappa \cdot t)), \quad V(0) = V_0 > 0,$$

on [0,T] using the same time steps.

(c) Since it is shown in Hanson (2008) that $4 \cdot \kappa \cdot \theta = \sigma_v^2$ implies an exact solution

$$\text{Vexact}(t) = \max(\text{Vdet}(t) + \text{Iexact}(t)\ ,\ 0),$$

where

$$\text{Iexact}(t) = \sigma_v \int_0^t \exp(-\kappa(s - t)) \cdot \sqrt{V(s)} \cdot dW_v(t).$$

It can be shown with some difficulty that the approximate update is

$$\text{Iexact}(t + \Delta t) = \exp(\Delta t) \cdot \left(\text{Iexact}(t) + \sigma_v \cdot \sqrt{V(t)} \cdot \Delta W_v(t)\right),$$

starting from $\text{Iexact}(0) = 0$ at $t_1 = 0$ and you need to simulate it too to computationally verify that the exact solution is correct.

(d) Plot the simulations of $V(t)$ of (ΔSV), $\text{Vdet}(t)$, $\text{Vexact}(t)$ and the error

$$\text{Verror}(t) = V(t) - \text{Vexact}(t)$$

on a single plot with appropriate legend. Also, print out the maximun of absolute value of $\text{Verror}(t)$.