MATHEMATICAL SCIENCE PRELIMINARY EXAMINATION

Monday, May 7, 2008 1:00-4:00pm

The Mathematical Science Preliminary examination covers the areas of Applied Optimal Control, Computational Finance, Mathematics of Fluid Dynamics, and Wave Propagation. Students elect to answer questions in only **two** of these areas.

This exam is based on questions from the areas: **Applied Optimal Control** and **Computational Finance**, **Mathematics of Fluid Dynamics**, and **Wave Propagation**. There are 4 questions in each area. Each question is worth 20 points. All questions in only two requested areas will be graded, but your score for the examination will be the sum of the scores of your best **FIVE** questions.

Use a separate answer booklet for each question, and do not put your name on the answer booklets, instead put the number that is on the envelope the exam was in. When you have completed the examination, insert all your answer booklets in the envelope provided. Then seal and print your name on the envelope.

Computational Finance

1. The price of a non-dividend paying asset in a **jump-diffusion market** is modeled by the following SDE:

$$dS(t) = S(t)(\mu_0 dt + \sigma_0 dW(t) + \nu_0 dP(t)), \; S(0) = S_0 > 0,$$

where μ_0, $\sigma_0 > 0$, ν_0, and $\lambda_0 > 0$ are real constants, where $W(t)$ is the standard diffusion process and $P(t)$ is the Poisson process with $\mathrm{E}[dP(t)] = \lambda_0 dt$.

   (a) Show that the explicit solution of the SDE is

$$S(t) = S_0 \exp((\mu_0 - \sigma_0^2/2)t + \sigma_0 W(t)) \cdot (1 + \nu_0)^{P(t)}.$$

   (b) If there is no jump process, i.e., $\nu_0 = 0$, then show that the distribution of the asset price satisfies the log-normal distribution

$$\Phi_{S(t)}(s) \equiv \mathrm{Prob}[S(t) \le s] = \Phi_n\left(\ln(s/S_0), (\mu_0 - \sigma_0^2/2)t, \sigma_0^2 t\right),$$

   where $\Phi_n(x, \mu, \sigma^2)$ is the normal distribution with mean μ and variance σ^2.

   (c) If there is a jump process, i.e., $\nu_0 \neq 0$, then show that

$$\Phi_{S(t)}(s) = \sum_{k=0}^{\infty} p_k(\lambda_0 t) \Phi_{W(t)}(M_k(s,t)),$$

   where $p_k(\lambda_0 t) = \mathrm{Prob}[P(t) = k]$ is the Poisson process distribution, $\Phi_{W(t)}(w)$ is the diffusion process distribution, and $M_k(s,t) \equiv (\ln(s/S_0) - (\mu_0 - \sigma_0^2/2)t - Q_0 k)/\sigma_0$, where $Q_0 \equiv \ln(1 + \nu_0)$.

2. Consider an option combination called a **Straddle** package associated with the following example:

   - Long (buy) a call option for $C_{\text{buy}} = \$3.50$ with an exercise price of $K = \$94$, and
   - Long (buy) a put option for $P_{\text{buy}} = \$6.50$ to sell the same underlying stock at the same time with the same exercise price of $K = \$94$, each for a same exercise date of $T = 6$ months.

   For the straddle in general and the example in particular:

   (a) Derive the general formula with general variables for the final value of the position at T.

   (b) Clearly sketch the net profit or loss graph versus the $S(T)$ value for the above example, illustrating the constituent calls that make up the spread and mark the break-even point or points, if any.

   (c) Discuss the relationship between exercise prices and between option prices needed in general variables for this position to be an effective strategy.

3. Consider an **American call option pricing** with strike price K and maturity at $T = 1$ year, when there is a single discrete dividend amount D_{ex} at the ex-dividend date $T_{ex} = 11/12$ of a year.

   (a) State, in words and formula, Fischer Black's approximation for simply estimating whether early exercise is the more profitable strategy. Justify all terms in the approximation.

   (b) Briefly describe the Roll-Geske-Whaley (RGW) method of using compound options to solve the problem. Discuss the advantages and disadvantages over Black's approximation.

   (c) Briefly discuss why, if there are earlier dividends then a final ex-dividend date like T_{ex}, that early exercise is still likely at T_{ex} if at all.

4. Derive the **risk-neutral** call option pricing, with strike price K and mature date T, in the a simple compound-jump-diffusion market environment for following dividendless asset price SDE:

$$dS(t) = S(t)(\mu_0 dt + \sigma_0 dW(t) + \nu_0 dP(t;\nu_0)),\ \ S(0) = S_0 > 0,$$

where μ_0, $\sigma_0 > 0$, and $\lambda_0 > 0$ are real constants, where $W(t)$ is the diffusion process and $P(t;\nu_0)$ is the compound Poisson process with jump-rate $\mathrm{E}[dP(t;\nu)] = \lambda_0 dt$ and IID jump-amplitude ν_0 with mean $\overline{\nu_0} \equiv \mathrm{E}[\nu_0]$ and density $\phi_{\nu_0}(q)$ given. Hull gives a risk-neutral recipe for diffusions, but here it is necessary to follow a more systematic procedure implied by Merton (1976) in his jump-diffusion paper.

   (a) Derive the solution for the final asset price $S(T)$.

   (b) Find the risk-neutral shift γ_d in the diffusion, i.e., $dW(t) - \gamma_d dt$, such that the total jump-diffusion mean is converted to the appropriate constant risk-less rate r_0, subject to the compound Poisson process being shifted to its mean-zero form, i.e., $\nu_0 dP(t;\nu_0) - \gamma_j dt$. Find the jump-risk γ_j along with total risk γ_d.

   (c) Let the risk-neutral (rn) call price, properly discounted, be defined by

$$C^{(rn)}(S_0, 0, K, T) \equiv e^{-r_0 T}\mathrm{E}^{(rn)}[\max[S(T) - K, 0] \mid S(0) = S_0].$$

   Show that, justifying all steps with reasons,

$$\begin{aligned} C^{(rn)}(S_0, 0, K, T) &= e^{-r_0 T} S_0 \sum_{k=0}^{\infty} p_k(\lambda_0 T) \int_{-\infty}^{\infty} dq \phi_{\nu_0}(q) \int_{W_k(T;S_0,K)}^{\infty} dw \phi_{W(T)}(w) \\ &\quad \cdot \left(\exp\left((r_0 - \sigma_0^2/2 - \overline{\nu_0}\lambda_0)T + \sigma_0 w + \mathcal{S}_k\right) - K/S_0\right), \end{aligned}$$

   where the ν_i are random samples of ν_0 and $\mathcal{S}_k \equiv \sum_{i=1}^{k} \ln(1 + \nu_i)$ is a partial sum. The formula for the diffusion-cutoff $W_k(T;S_0,K)$ must be derived. Discuss what further transformations are needed to form jump-modified Black-Scholes call coefficients.