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Optimal Consumption and Portfolio Policies for Important Jump Events: Modeling and Computational Considerations*

F. B. Hanson
Laboratory for Advanced Computing
University of Illinois at Chicago
851 Morgan St.; M/C 249
Chicago, IL 60607-7045, USA
E-Mail: hanson@math.uic.edu
URL: http://www.math.uic.edu/~hanson/

and

J. J. Westman
Department of Mathematics
University of California
Box 951555
Los Angeles, CA 90095-1555, USA
E-Mail: jwestman@math.ucla.edu
URL: http://www.math.ucla.edu/~jwestman/

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Abstract

While the volatility of portfolios are often modeled by continuous Brownian motion processes, discontinuous jump processes are more appropriate for modeling important external events that significantly affect the prices of financial assets. Here the discontinuous jump processes are modeled by state and control dependent compound Poisson processes, such that the random jumps come at the times of a pure Poisson process with jump amplitudes that are randomly distributed. The optimal consumption and investment portfolio policy formulation is in terms of stochastic differential equations with optimal discounted utility objectives. This paper was motivated by a recent paper of Rishel (1999) concerning portfolio optimization when prices are dependent on external events. However, the model has been significantly generalized for realistic computational considerations.

1. Introduction

While much of the continuous time models of financial markets have been based upon continuous sample path geometric Brownian motion processes, Merton [14] applied discontinuous sample path Poisson processes, along with Brownian motion processes, i.e., jump diffusions, to the problem of pricing options. He derived several extensions of the already classical diffusion theory of Black and Scholes [1] applying minimizing the portfolio variance techniques to jump diffusion models similar to those techniques used to derive the classic Black and Scholes formulae. Earlier, Merton [13] treated optimal consumption and investment portfolio with either geometric Brownian motion or Poisson noise, illustrated explicit solutions for constant risk-aversion in either relative and absolute forms. In [12], Merton also examined these constant risk-aversion cases. In [9], Karatzas, Lehoczky, Sethi and Shreve pointed out that it is necessary to enforce non-negativity feasibility conditions on both wealth and consumption, deriving formally explicit solutions from a consumption-investment dynamic programming models with a time-to-bankruptcy horizon, that qualitatively correct the results of Merton [13]. Sethi and Taksar [18] directly present corrections to certain formulae

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Merton's finite horizon consumption-investment model [13]. Merton [15] revisited the problem in the 6th chapter of his book, correcting his earlier work by adding an absorbing boundary condition at zero wealth and using other techniques.

Wilmott's [23] presents a good discussion on hedging with jump diffusion models in finance. Lipton-Lifshitz [11] presents a good discussion of predictability and unpredictability, mainly for foreign exchange applications, but is applicable to other financial applications as well.

Recently, Rishel [17] introduced a optimal portfolio model for stock prices dependent on deterministic scheduled and stochastic unscheduled jump external events based on optimal stochastic control theory. The jumps can affect both the stock prices directly or indirectly through parameters. The deterministic jumps are deterministic only in the timing of the scheduled events, but the magnitude of the jumps is random. Rishel's paper and related bioeconomic and manufacturing research of Hanson and co-workers [7, 20, 21, 22] are the motivation for the present paper. Here our formulation is a modification on Rishel's paper, with heavier reliance on stochastic differential equations, constrained control, more general utility objectives, generalized functions, and random Poisson measure. Many of the modifications make the model more realistic and computational feasible. Major computations for the model will be the subject of another paper. The approach here is more toward applied modeling and computational feasibility rather than purely mathematical theorem statement and proving.

The paper is arranged as follows. In Section 2., the stochastic differential equation model for the underlying risk-free and risky assets are formulated in terms of state and control dependent marked Poisson processes with diffusion processes used for the less important background events. In Section 3., the wealth equations are formulated for the portfolio. In Section 4., the portfolio optimization problem is formulated and the subsequent partial differential equation of stochastic dynamic programming is derived from the generalized Itô chain rule. The formulation of the solution for the power utilities example in Section 5. is significantly reduced to much simpler equations with many explicit terms, although some implicit terms remain, especially those Poisson related, but these equations are suitable for computation. Further computational considerations are discussed in Section 6.

2. State and Control Dependent Asset Models

In this paper we are interested in selecting a financial portfolio from a risk-free asset or bond and a number of risky assets or stocks. Let the bond earn a fixed rate r of interest such that its price $B(t)$ at time t satisfies the deterministic dynamical process,

$$dB(t) = rB(t)dt, \quad B(0) = B_0. \tag{1}$$

The coefficient r also could be made time dependent, e.g., jumping at times of events of the change in federal rates.

Let $S_i(t)$ denote the price of the ith stock which satisfies the Markov geometric jump-diffusion stochastic differential equation,

$$dS_i(t) \quad = \quad S_i(t) \left[\mu_i(\vec{A}(t))dt + \sum_{j=1}^{M} \sigma_{i,j}(\vec{A}(t))dZ_j(t) + dP_i(t) \right. \tag{2}$$

$$+\sum_{\ell=1}^{N_2} J_{2,i}(t,\vec{j}_{2,\ell};\vec{A}(t))\delta_R(t-T_{2,\ell})dt\Bigg], \quad S_i(0)=S_{0,i},$$

for $i=1,2,\ldots,N$ stocks and with parameter vector $\vec{A}(t)=(A_1,A_2)$. First the terms in (2) will be briefly identified, but will be more thoroughly described later. The $\mu_i(\vec{a})=\mu_i(a_1,a_2)$ denote the mean appreciation rate for the ith stock as a function of unscheduled a_1 and scheduled a_2 parameters; the $\sigma_{i,j}(\vec{a})$ are the volatilities for the ith stock due to the jth continuous, Brownian motion processes $Z_j(t)$ for $j=1,2,\ldots,M$; the $dP_i(t)$ denote the discontinuous, random, space-time Poisson processes representing important unscheduled events; and the $\sum_{\ell=1}^{N_2} J_{2,i}\delta_R(t-T_{2,\ell})dt$ represent the corresponding scheduled events with deterministic jump times, except have random amplitudes like the space-time Poisson processes.

The $Z_j(t)$, for $j=1,2,\ldots,M$, denote independent, standard Brownian motion processes and $P_i(t)$ denotes the ith component of a marked Poisson process. The continuous sample path processes $Z_j(t)$ model the less important background random events that affect the market, while the discontinuous sample path processes $dP_i(t)$ model the rare, important events that lead to large fluctuations in risk sensitive market assets. The space-time differential Poisson processes $dP_i(t)$ are related to Poisson random measure or space-time Poisson processes, $\mathcal{P}(dt,d\widehat{j}_1)$, (see Itô [8], Gihman and Skorohod [4], or Snyder and Miller [19]):

$$dP_i(t)=\int_{\mathcal{J}_1} J_{1,i}(t,\widehat{j}_1;\vec{A})\mathcal{P}(dt,d\widehat{j}_1), \tag{3}$$

for $i=0,1,\ldots,N$, where $J_{1,i}$ is the ith Poisson jump amplitude function corresponding to the ith stock price, $\widehat{j}_1=(j_{1,0},\vec{j}_1)=[j_{1,i-1}]_{(N+1)\times 1}$ is the $(N+1)$-dimensional random mark vector on the mark space $\mathcal{J}_1$. Each time the constituent Poisson counting process has a jump signifying an unscheduled event, a random mark vector $\widehat{j}_1$ is generated which in turn generates the value of the vector jump amplitude $\widehat{J}_1=(J_{1,0},\vec{J}_1)(t,\widehat{j}_1;\vec{A})$, resulting in the jump in the unscheduled parameter A_1 from $J_{1,0}$ and in the jump in stock price S_i from $J_{1,i}$ for $i=1,2,\ldots,N$, respectively. The component $dP_i(t)$ of the Poisson driven process has the expectation:

$$E[dP_i(t)]=\lambda(t)\int_{\mathcal{J}_1} J_{1,i}(t,\widehat{j}_1;\vec{A})\phi_1(\widehat{j}_1)d\widehat{j}_1 dt \equiv \lambda(t)E[J_{1,i}]dt, \tag{4}$$

for $i=0,1,\ldots,N$, where $\lambda(t)$ is the rate for the common Poisson counting process, and $\phi_1(\widehat{j}_1)$ is the joint density of the amplitude marks. Assuming component-wise independence, $dP_i(t)$ has covariance given [19] by

$$\mathrm{Var}[dP_i(t)]=\lambda(t)E[J_{1,i}^2]dt, \tag{5}$$

for $i=1,2,\ldots,N$. Given that there is a jump at $T_{1,\ell}$, the stock price $S_i(t)$ jump magnitude is

$$[S_i](T_{1,\ell})\equiv S_i(T_{1,\ell}^+)-S_i(T_{1,\ell}^-)=J_{1,i}(T_{1,\ell},\widehat{j}_1;\vec{A})S_i(T_{1,\ell}^-),$$

for $\ell=1,2,\ldots,N$, assuming that the jump amplitude is continuous in t. A simple Poisson jump amplitude model could have $J_{1,i}(t,\widehat{j}_1;\vec{a})=j_{1,i}$, i.e., the jump amplitude vector being the same as the mark vector.

The unscheduled event parameter $A_1(t)$ is generated by the same space-time Poisson process $\mathcal{P}$ as above, reserving the components $J_{1,0}$ of the jump and $j_{1,0}$ of the mark for $A_1(t)$, so that the jump in $A_1(t)$ is generated at the same time by the single underlying Poisson counting process of the unscheduled event as it is for the stocks in (2) at rate

$\lambda(t)$ for unscheduled events,

$$dA_1(t) = A_1(t)dP_0(t),$$

where $dP_0(t)$ is given by (3) when $i = 0$, with conditional expected jump amplitude of the parameter differential

$$E[dA_1(t)|\vec{A}(t) = \vec{a}] = \lambda(t)a_1 E[J_{1,0}|\vec{A}(t) = \vec{a}]dt,$$

where $E[J_{1,0}]$ is the average jump amplitude of the 0th component of the space-time Poisson process. Again, a relative jump size is used here, rather than an absolute jump amplitude.

The last term on the right hand side of (2) models the jumps resulting from scheduled events at times $T_{2,\ell}$, for $\ell = 1, 2, \ldots, N_2$ and relative jump amplitude of $J_{2,i}(t, \vec{j}_{2,\ell}; \vec{A}(t))$ causing $S_i(t)$ to jump by $J_{2,i}S_i(T_{2,\ell}^-)$, assuming $T_{2,\ell} < T_{2,\ell+1}$, where $\vec{j}_{2,\ell} = [j_{2,i,\ell}]_{N\times 1}$ is the random mark vector for scheduled jumps while $\vec{J}_2$ is the corresponding relative jump amplitude, such that

$$\text{Prob}\left[j_{2,i,\ell} \in (j_{2,i}, j_{2,i} + dj_{2,i})|A_2(T_{2,\ell}^-) = a_{2,\ell}^-\right] = \phi_{2,i}(j_{2,i}; a_{2,\ell}^-)dj_{2,i}, \tag{6}$$

where

$$\phi_{2,i}(\overline{j}_{2,i}; a_2) = \int_{\mathcal{J}_2} \delta(j_{2,i} - \overline{j}_{2,i})\phi_2(\widehat{j_2}; a_2)d\widehat{j_2}$$

is a proper density function, which may include discrete distributions through generalized functions, with $\delta(x)$ denoting the usual Dirac delta function and $\widehat{j_2} = [j_{2,i-1}]_{(N+1)\times 1}$. The generalized function symbol $\delta_R(t - T_{2,\ell})$ denotes a *right continuous delta function* defined by

$$\int_{-\infty}^{\infty} f(t)\delta_R(t - T_\ell) = f(T_\ell^-),$$

for some right-continuous function f, compatible with the right continuity (continuity from the right) of the Poisson process. Unlike the Dirac delta function, $\delta_R(t-T_{2,\ell})$ is a bounded step function embodied in its constructive definition as the difference of step functions,

$$\delta_R(t - T_{2,\ell})dt = H_R(t + dt - T_{2,\ell}) - H_R(t - T_{2,\ell}),$$

for infinitesimal dt, where $H_R(t-T_{2,\ell})$ is the right-continuous unit step function that characterizes the simple Poisson counting process $\sum_{\ell=1}^{\infty} H_R(t - T_{2,\ell})$. Thus, the scheduled jump amplitude S_i at $T_{2,\ell}$ is

$$[S_i](T_{2,\ell}) \equiv S_i(T_{2,\ell}^+) - S_i(T_{2,\ell}^-) = J_{2,i}(T_{2,\ell}^-, j_{2,\ell}; \vec{A}(T_{2,\ell}^-))S_i(T_{2,\ell}^-),$$

due to non-anticipating, right-continuity and such that stock i jumps from $S_i(T_{2,\ell}^-)$ to $J_{2,i}S_i(T_{2,\ell}^-)$ at the scheduled jump time $T_{2,\ell}$. It is further assumed that the final scheduled jump at T_{2,N_2} takes place before the terminal time T, i.e., $T_{2,N_2} < T < T_{2,N_2+1}$. These scheduled jumps affect the market due to events such as changes in monetary policy or announcements of labor statistics or other economic announcements or eminent labor strikes, although the magnitude of the jumps can be random, as described by Rishel [17]. A recent (February 17, 2000) example is the semi-annual economic report of Chairman Alan Greenspan of the Federal Reserve Board to Congress that concerned the raising of interest rates and other matters followed the next day (February 18, 2000) by a "double witching day" in which there was a simultaneous expiration of contracts on stock options and stock indices.

For the continuous portion of the sample paths, the non-anticipating mean appreciation rate is $\mu_i(\vec{A}(t))$ and the squared volatility is $\sum_{j=1}^{M}\sigma_{i,j}^2(\vec{A}(t))$, relative to and conditioned on the current stock price $S_i(t)$, i.e., in absence of the scheduled or random jump events. The vector $\vec{A}(t)=(A_1(t),A_2(t))$ represents parametric arguments of the mean appreciation rate μ_i, volatility $\sigma_{i,j}$ and scheduled jump amplitudes. The parameter $A_2(t)$ is assumed to have jumps at the same times as that of the scheduled events,

$$dA_2(t) \;=\; A_2(t)\sum_{\ell=1}^{N_2} J_{2,0}(t,j_{2,0,\ell},\vec{A}(t))\delta_R(t-T_{2,\ell})dt, \tag{7}$$

$$[A_2](T_{2,\ell}) \;=\; A_2(T_{2,\ell}^-)J_{2,0}(T_{2,\ell}^-,j_{2,0,\ell},\vec{A}(T_{2,\ell}^-)), \tag{8}$$

$$E_{J_{2,0}}[A_2(T_{2,\ell}^+)|A_2(T_{2,\ell}^-)=a_{2,\ell}^-] \;=\; (1+\overline{J}_{2,0})a_{2,\ell}^-, \tag{9}$$

where the mean jump distribution $\overline{J}_{2,0}$ for A_2 is likely different from scheduled mean jump distribution $\overline{J}_{2,i}$ for S_i for $i=1,2,\ldots,N$ stocks. Here, a relative jump size is used, rather than an absolute size in Rishel [17], i.e., geometric or multiplicative noise is used here rather additive noise.

Our model for the underlying assets is the similar to that of Rishel [17], except that more general distributions are used here for the appreciation, volatility and unscheduled jump parameters, rather than the discrete random states used in [17]. Also, space-time Poisson processes are used extensively in the model.

3. Portfolio Wealth Equation

Let $W(t)$ be the portfolio wealth process for a portfolio at time t that includes a risk-free bond asset at price $B(t)$ and the N risky stocks $S_i(t)$. Let $U_i(t)$ be the fraction of the wealth $W(t)$ invested in the ith asset at time t for $i=1,2,\ldots,N$ and $U_0(t)$ will denote the fraction invested in bonds at time t, so that

$$U_0(t)+\sum_{i=1}^{N}U_i(t)=1, \tag{10}$$

which serves as the defining constraint for the bond fraction $U_0(t)$ in terms of the stock fraction vector $\vec{U}(t) = [U_i(t)]_{N\times 1}$. Along with consumption of capital, the stock investment fractions will comprise the components of the control vector, $\vec{U}(t)=[U_i(t)]_{N\times 1}$, for this problem, such that $U_i(t)$ can take on arbitrary real values if $i>0$ in theory, since the fraction of stock i can be negative if the stock is sold short at time t in anticipation of a drop in prices making it profitable to buy back later, while the fraction invested in bonds can be negative if money is borrowed on the bond and invested in stock i with $i>0$ (i.e., the sum over the stock fractions in (10) can exceed unity and thus are unbounded above, in theory). However, for practical reasons, the stock fractions must be bounded or limited since borrowing and short selling would be limited. Further, if the jump model leads to singular control calculations, then the control space would need to be bounded. Thus, the control space will be assumed bounded, for example, component-wise constraints, $U_{\min,i}\le u_i\le U_{\max,i}$, with $U_{\min,i}\le 0$ and $U_{\max,i}>0$, specified.

Thus the wealth $W(t)=\sum_{i=0}^{N}U_i(t)W(t)$ at time t and the dynamics of the amount $U_i(t)W(t)$ invested in instrument i at time t satisfies (1) for the bond when $i=0$ and (2) for the ith when $i>0$ stock. The wealth less consumption, $C(t)$, satisfies the SDE,

$$dW(t) \;=\; -C(t)dt+W(t)\Bigg[rdt+\vec{U}^\top(t)\Bigg\{(\vec{\mu}(\vec{A}(t))-r\vec{1})dt+\sigma(\vec{A}(t))d\vec{Z}(t)+d\vec{P}(t) \tag{11}$$

$$+ \sum_{\ell=1}^{N_2} \vec{J}_2(t, \vec{j}_{2,\ell}; \vec{A}(t))\delta_R(t - T_{2,\ell})dt \Bigg\} \Bigg],$$

with matrix-vector notation: $\vec{U}^\top(t) = [U_j(t)]_{1\times N}$ denotes the transpose of $\vec{U}(t)$, $\vec{1} = [1]_{N\times 1}$, $\vec{\mu}(\vec{a}) = [\mu_i(\vec{a})]_{N\times 1}$, $\sigma(\vec{a}) = [\sigma_{i,j}(\vec{a})]_{N\times M}$, $d\vec{Z}(t) = [dZ_i(t)]_{M\times 1}$, $d\vec{P}(t) = [dP_i(t)]_{N\times 1}$ (the 0th component for associated with the random unscheduled jump parameter $A_1(t)$ is not directly included in the wealth equation), and $\vec{J}_k = [J_{k,i}]_{N\times 1}$, $k = 1$ for unscheduled jumps and $k = 2$ for scheduled jumps. Note that the jump in wealth is given by

$$[W](T_{k,\ell}) \equiv W(T^+_{k,\ell}) - W(T^-_{k,\ell}) = \sum_{i=1}^{N} U_i(T^-_{k,\ell})J_{k,i}(T_{k,\ell}, \vec{j}_{k,\ell}; \vec{A}(t))W(T^-_{k,\ell}), \tag{12}$$

at each jump time $t = T_{k,\ell}$, for $\ell = 1, 2, 3, \ldots$ when $k = 1$ for unscheduled jumps or $\ell = 1, 2, \ldots, N_2$ when $k = 2$ for scheduled jumps.

4. Consumption and Portfolio Optimization for Expected Utility Problem

Let $\mathcal{U}_f(w; \vec{a})$ be the utility function of final wealth as well as of the vector parameter $\vec{a}$, and $\mathcal{U}(c)$ be the instantaneous utility of consumption for the investor. Suppose the investor consumes $c = C(t)$ at time t and ends up with wealth $w = W(T)$ at the final time T and that the investor seeks to maximize the conditional expected, *current value* at t of the discounted utility of the terminal wealth and instantaneous consumption, i.e.,

$$\begin{aligned} V(t, w, \vec{u}, c; \vec{a}) \quad = \quad & E\Bigg[e^{-\beta(T-t)}\mathcal{U}_f(W(T); \vec{a}) + \int_t^T e^{-\beta(\tau-t)}\mathcal{U}(C(\tau))d\tau \\ & \Bigg| W(t) = w, \vec{U}(t) = \vec{u}, C(t) = c; \vec{A}(t) = \vec{a}\Bigg], \end{aligned} \tag{13}$$

by selecting the maximizing portfolio policies $\vec{U}(t)$ and consumption $C(t)$, assuming the wealth process $W(t)$ satisfies the stochastic dynamics specified by (11). Discounting is used here to account for opportunity costs due to alternative investments, in contrast to [17]. Here, β is the real (nominal less inflation) discount rate, assumed fixed here, but, for example, could be made to jump with the announced announced changes in the federal funds discount rate. The utility functions $\mathcal{U}(c)$ and $\mathcal{U}_f(w; \vec{a})$ are assumed to be increasing concave functions, i.e., $\mathcal{U}'(c) > 0$ and $\mathcal{U}''(c) < 0$, for example. The differences from Rishel's [17] paper are that parameter vector $\vec{a}$ is included in the terminal wealth utility making $\vec{a}$ genuinely included in the model and also the cumulative discounted running utility for consumption of wealth is part of the objective.

Let the optimal expected utilities of the portfolio be $v(t, w; \vec{a}) = \max_{\{\vec{u},c\}[t,T)}[V(t, w, \vec{u}, c; \vec{a})]$, subject to the non-negative feasibility conditions on consumption $C(t) \geq 0$ and on wealth $W(t) \geq 0$ making zero wealth an absorbing state to avoid the possibility of arbitrage [15], where the vector parameter $\vec{a} = (a_1, a_2)$ forms an extension of the state space from the wealth state w. Due to the non-anticipating properties of the Markov and deterministic processes, Bellman's Principle of optimality is valid in infinitesimal form as

$$v(t, w; \vec{a}) = \max_{\{\vec{u},c\}[t,t+dt)} \left[E_{[t,t+dt)}\left[\mathcal{U}(c)dt + (1 - \beta dt)v(t + dt, w + dW(t); \vec{a} + d\vec{A}(t))\right]\right], \tag{14}$$

for $0 \leq t < T$, subject to non-negative consumption $C(t) \geq 0$, zero wealth absorbing boundary condition

$$v(t, 0^+; \vec{a}) = e^{-\beta(T-t)}\mathcal{U}_f(0; \vec{a}) + \mathcal{U}(0)(1 - e^{-\beta(T-t)})/\beta, \tag{15}$$

to account for the non-negative wealth condition $W(t) \geq 0$ assuming that consumption must be zero when wealth is zero, and subject to the bequest or terminal wealth condition

$$v(T, w; \vec{a}) = \mathcal{U}_f(w; \vec{a}). \tag{16}$$

Assuming that $v(t, w; \vec{a}) = v(t, w; a_1, a_2)$ is continuously differentiable in t, twice continuously differentiable in w and continuous in the vector parameter $\vec{a}$ between scheduled jumps, then stochastic dynamic programming equations between scheduled jumps (see Kushner [10], Itô [8], Gihman and Skorohod [5], Snyder and Miller [19] for the less familiar Poisson driven terms) is

$$\begin{aligned} 0 \;=\;& v_t(t, w; \vec{a}) - \beta v(t, w; \vec{a}) + \max_{\{\vec{u}, c\}} \Big[\mathcal{U}(c) + \Big((r + \vec{u}^\top(\vec{\mu}(\vec{a}) - r\vec{1}))w - c\Big) v_w(t, w; \vec{a}) \\ &+ \frac{1}{2}\vec{u}^\top \sigma(\vec{a})\sigma^\top(\vec{a})\vec{u}w^2 v_{ww}(t, w; \vec{a}) \\ &+ \lambda(t)\int_{\mathcal{J}_1}\Big[v(t, (1 + \vec{u}^\top \vec{J}_1(t, \widehat{j}_1; \vec{a}))w; (1 + J_{1,0}(t, \widehat{j}_1; \vec{a}))a_1, a_2) - v(t, w; \vec{a})\Big]\phi_1(\widehat{j}_1)d\widehat{j}_1\Big], \\ =\;& v_t(t, w; \vec{a}) - \beta v(t, w; \vec{a}) + \mathcal{U}(c^*) + \Big((r + (\vec{u}^*)^\top(\vec{\mu}(\vec{a}) - r\vec{1}))w - c^*\Big) v_w(t, w; \vec{a}) \\ &+ \frac{1}{2}(\vec{u}^*)^\top \sigma(\vec{a})\sigma^\top(\vec{a})\vec{u}^* w^2 v_{ww}(t, w; \vec{a}) \\ &+ \lambda(t)\int_{\mathcal{J}_1}\Big[v(t, (1 + (\vec{u}^*)^\top \vec{J}_1(t, \widehat{j}_1; \vec{a}))w; (1 + J_{1,0}(t, \widehat{j}_1; \vec{a}))a_1, a_2) - v(t, w; \vec{a})\Big]\phi_1(\widehat{j}_1)d\widehat{j}_1, \end{aligned} \tag{17}$$

where $\vec{u}^* = \vec{u}^*(t, w; \vec{a})$ and $c^* = c^*(t, w; \vec{a})$ are the optimal controls if they exist, v_w and v_{ww} are the partial derivatives with respect to wealth, when $T_{2,\ell} < t < T_{2,\ell+1}$, or in jump time notation $T^+_{2,\ell} \leq t < T^-_{2,\ell+1}$, for $\ell = N_2 \ldots, 1, 0$ by backward counting with $T_{2,N_2+1} = T^-_{2,N_2+1} \equiv T$ finally and $T_{2,0} = T^+_{2,0} \equiv 0$ initially.

At the scheduled jumps, backward from $t = T^+_{2,\ell}$ to $t = T^-_{2,\ell}$, the value function jumps due to the fact that the scheduled jump times are not averaged over as are the unscheduled Poisson jumps (see [17] for a somewhat different formulation) and takes its value from the jump (8) in A_2 and (12) in wealth,

$$v(T^-_{2,\ell}, w; \vec{a}) = \max_{\{\vec{u}^-_{2,\ell}\}}\left[\int_{\mathcal{J}_2} v(T^+_{2,\ell}, (1 + (\vec{u}^-_{2,\ell})^\top \vec{J}_2(T^-_{2,\ell}, \vec{j}_2; \vec{a}))w; a_1, (1 + J_{2,0}(T^-_{2,\ell}, j_{2,0}; \vec{a}))a_2)\phi_2(\widehat{j}_2)d\widehat{j}_2\right], \tag{18}$$

for $\ell = 1, 2, \ldots, N_2$, where $\vec{u}^-_{2,\ell} \equiv \vec{u}(T^-_{2,\ell})$ since $W(T^+_{2,\ell}) = (1 + \vec{u}^\top(T^-_{2,\ell})\vec{J}_2)W(T^-_{2,\ell})$, and $\widehat{j}_2 = (j_{2,0}, \vec{j}_2)$, and the right continuity property along with the instantaneous jump property has been used. Since dynamic programming is a backward formulation in time, (18) is a condition for $v(T^-_{2,\ell}, w; \vec{a})$ rather than $v(T^+_{2,\ell}, w; \vec{a})$ given by (17), although the argument of the maximum is the optimal control $u^*(T^-_{2,\ell}, w; \vec{a})$, while there is no corresponding value for the optimal consumption at $T^+_{2,\ell}$ in this equation. This jump condition illustrates that deterministic jumps are more difficult to treat than Poisson jumps. Here, the bond fraction has been eliminated by (10) in favor of the stock fractions, i.e., $u_0 = 1 - \sum_{i=1}^N u_i = 1 - \vec{u}^\top\vec{1}$, where $\vec{1} \equiv [1]_{N\times 1}$ is the summing vector.

If the unconstrained maximum in (17) is attained by the regular controls $\vec{u}_{\rm reg}(t, w; \vec{a})$ and $c_{\rm reg}(t)$, then on $T_{2,\ell-1} < t < T_{2,\ell}$ they implicitly satisfy the equations

$$\mathcal{U}'(c_{\rm reg}(t, w; \vec{a})) = v_w(t, w; \vec{a}), \tag{19}$$

$$\begin{aligned} w^2 v_{ww}(t, w; \vec{a})\sigma(\vec{a})\sigma^\top(\vec{a})\vec{u}_{\rm reg}(t, w; \vec{a}) = -w v_w(t, w; \vec{a})(\vec{\mu}(\vec{a}) - r\vec{1}) \\ -\lambda(t)w\int_{\mathcal{J}_1}\vec{J}_1(t, \widehat{j}_1; \vec{a})v_w(t, (1 + \vec{u}^\top_{\rm reg}(t, w; \vec{a})\vec{J}_1(t, \widehat{j}_1; \vec{a}))w; (1 + J_{1,0}(t, \widehat{j}_1; \vec{a}))a_1, a_2)\phi_1(\widehat{j}_1)d\widehat{j}_1, \end{aligned} \tag{20}$$

for the optimal consumption and portfolio policies with respect to the terminal wealth and instantaneous consumption utilities (13).

At the scheduled jumps, $t = T_{2,\ell}$, the portfolio policy must jump when the optimal portfolio value jumps, but it may be practical to bring policy constraints into play since the first derivative critical condition,

$$w \int_{\mathcal{J}_2} \vec{J}^{\,-}_{2,\ell} v_w(T^{+}_{2,\ell}, (1 + (\vec{u}^{\,-}_{\text{reg},2,\ell})^{\top} \vec{J}^{\,-}_{2,\ell})w; a_1, (1 + J^{-}_{2,0,\ell})a_2)\phi_2(\hat{j}_2)d\hat{j}_2 = \vec{0}, \tag{21}$$

for the argument the maximum in the optimal jump condition (18), may not have a regular or unconstrained solution for $\vec{u}_{\text{reg}}$, especially if the value policy derivative, v_w, is nonvanishing. The mathematically ideal infinite investment fraction control domain may not be practical, so that a finite control domain is considered. Here, $\vec{u}^{\,-}_{\text{reg},2,\ell} \equiv \vec{u}_{\text{reg}}(T^{-}_{2,\ell}, w; \vec{a})$, $\vec{J}^{\,-}_{2,\ell} \equiv \vec{J}_2(T^{-}_{2,\ell}, \vec{j}_2; \vec{a})$ and $J^{-}_{2,0,\ell} \equiv J_{2,0}(T^{-}_{2,\ell}, j_{2,0}; \vec{a})$.

5. Constant Relative Risk-Aversion Utility Example

When the utility functions appearing in the objective functional (13) are power functions,

$$\begin{aligned} \mathcal{U}(c) &= c^{\gamma}/\gamma, \;\; c \geq 0, \;\; 0 < \gamma < 1, \\ \mathcal{U}_f(w; \vec{a}) &= \mathcal{U}(w)\mathcal{U}_1(a_1)\mathcal{U}_2(a_2), \;\; w \geq 0, \\ \mathcal{U}_k(a_k) &= |a_k|^{\gamma_k}, \;\; \gamma_k \neq 0, \;\; k = 1, 2. \end{aligned} \tag{22}$$

Using arbitrary powers of consumption and wealth means that, in order to enforce real values on the utility functions, consumption and wealth must be non-negative. This is the case of iso-elastic marginal utility or constant relative risk-aversion (CRRA), since the elasticity of the marginal utility or relative risk-aversion (RRA) is the ratio of the marginal rate to the average rate for the marginal utility. This also a special subcase of the Hyperbolic Absolute Risk Aversion (HARA) utility case treated by Merton [12, 13].

With these power utility functions, a good guess for the form of the solution is by partial multiplicative separation of variables,

$$v(t, w; \vec{a}) = \mathcal{U}_f(w; \vec{a})v_0(t; \vec{a}), \tag{23}$$

where the separated time function $v_0(t; \vec{a})$ is to be determined. The absorbing boundary condition (15) is nominally satisfied by (23) since $\mathcal{U}_f(0; \vec{a}) = 0$ by (22). Substitution of the solution form (23), yields an explicit form for the regular control consumption values using (19),

$$c_{\text{reg}}(t, w; \vec{a}) = wq_2(\vec{a})/v_0^{1/(1-\gamma)}(t; \vec{a}), \tag{24}$$

provided $v_0(t; \vec{a}) \neq 0$, where $q_2(\vec{a}) \equiv 1/\left[\mathcal{U}_1(a_1)\mathcal{U}_2(a_2)\right]^{1/(1-\gamma)}$, provided $a_k \neq 0$ for each k. Also, an implicit form for the stock fractions using (20) ,

$$\vec{u}_{\text{reg}}(t; \vec{a}) \;\; = \;\; \frac{1}{1-\gamma}(\sigma\sigma^{\top})^{-1}(\vec{a}) \left[\vec{\mu}(\vec{a}) - r\vec{1} + \frac{\lambda(t)}{\gamma}\vec{I}_1{}'(\vec{u}_{\text{reg}}(t; \vec{a}), t; \vec{a})\right], \tag{25}$$

where

$$\vec{I}_1{}'(\vec{u},t;\vec{a}) \;\equiv\; \int_{\mathcal{J}_1} \vec{J}_1(t,\widehat{j}_1;\vec{a})\frac{\mathcal{U}(1+\vec{J}_1^\top(t,\widehat{j}_1;\vec{a})\vec{u})}{(1+\vec{J}_1^\top(t,\widehat{j}_1;\vec{a})\vec{u})}\mathcal{U}_1(1+J_{1,0})\psi(t,\vec{j}_1;\vec{a})\phi_1(\widehat{j}_1)d\widehat{j}_1, \tag{26}$$

$$\psi(t,\vec{j}_1;\vec{a}) \;\equiv\; \frac{v_0(t;(1+J_{1,0}(t,\vec{j}_1;\vec{a})a_1,a_2)}{v_0(t;a_1,a_2)}, \tag{27}$$

and provided that the diffusion matrix, $\sigma(\vec{a})\sigma^\top(\vec{a})$, is invertible. Note that $\vec{u}_{\text{reg}}(t;\vec{a})$ is independent of the wealth w, which is a crucial property needed for partial separability.

Substitution of the power solution ansatz (23) and the regular controls in (24-25) into the stochastic dynamic programming equation (17), leads to an ordinary differential equation depending on the vector parameter $\vec{a}$ and this equation can be viewed as an implicit Bernoulli equation with variable coefficients for sufficiently small parameter values,

$$0 \;=\; v_0'(t;\vec{a})+(1-\gamma)\left(q'_{1,\text{reg}}(t;\vec{a})v_0(t;\vec{a})+q_2(\vec{a})v_{0,\text{reg}}^{\frac{\gamma}{\gamma-1}}(t;\vec{a})\right), \tag{28}$$

$$\begin{aligned} q'_{1,\text{reg}}(t;\vec{a}) \;&\equiv\; \frac{1}{1-\gamma}\Big[-\beta+\gamma\left(r+\vec{u}_{\text{reg}}^\top(t;\vec{a})(\vec{\mu}(\vec{a})-r\vec{1})\right)+\lambda(t)(I_1(\vec{u}_{\text{reg}}(t;\vec{a}),t;\vec{a})-1) \\ &\quad -\frac{\gamma(1-\gamma)}{2}\left[\vec{u}_{\text{reg}}^\top(t;\vec{a})(\sigma(\vec{a})\sigma^\top(\vec{a}))^{-1}\vec{u}_{\text{reg}}(t;\vec{a})\right]\Big], \end{aligned} \tag{29}$$

$$I_1(\vec{u},t;\vec{a}) \;\equiv\; \gamma\int_{\mathcal{J}_1}\mathcal{U}(1+\vec{J}_1^\top(t,\widehat{j}_1;\vec{a})\vec{u})\mathcal{U}_1(1+J_{1,0}t,\widehat{j}_1;\vec{a}))\psi(t,t,\widehat{j}_1;\vec{a})\phi_1(\widehat{j}_1)d\widehat{j}_1, \tag{30}$$

for t on $[T^+_{2,\ell-1},T^-_{2,\ell})$ for $\ell=N_2+1\ldots,2,1$ subintervals with $T_{2,0}\equiv 0$ and $T_{2,N_2+1}\equiv T$. The formula (26) defining $I_1{}'(\vec{u},t;\vec{a})$ is the control gradient of $I1(\vec{u},t;\vec{a})$. In the presence of control constraints, constrained perturbations of $q_{1,\text{reg}}(t;\vec{a})$, upon replacing the unconstrained $\vec{u}_{\text{reg}}$ with the constrained optimal $\vec{u}^*$, force iterative perturbations on $v_{0,\text{reg}}(t;\vec{a})$ to yield approximations of the constrained, scaled optimal value $v_0(t;\vec{a})$. The advantage is that the perturbation is still independent of the state of the wealth.

The partial separability assumption where $\vec{a}$ still appears in the time function $v_0(t;\vec{a})$ is due to the $\vec{a}$-dependence of the coefficients of the variance $\sigma(\vec{a})$, the mean return $\mu(\vec{a})$, and the utilities $\mathcal{U}_k(a_k)$. If these coefficients were independent of $\vec{a}$, then the time function could be heuristically replaced by just $v_0(t)$. Otherwise, the implicit dependence on $\psi(t,\vec{j}_1;\vec{a})$ in Eqs. (26,30) will require iterative, interpolation, or other approximate solutions.

The implicit Bernoulli equation (28) can be formally transformed to an easily integrable linear differential equation by the change of variables $\theta(t)=v_0^{1-\gamma/(\gamma-1)}(t;\vec{a})=v_0^{1/(1-\gamma)}(t;\vec{a})$,

$$0=\theta'(t)+q'_{1,\text{reg}}(t;\vec{a})\theta(t)+q_2(\vec{a}), \tag{31}$$

which has the general solution which easily be converted to the general solution for the desired time function,

$$v_0(t;\vec{a})=\theta^{1-\gamma}(t;\vec{a})=\left[e^{-q_{1,\text{reg}}(t;\vec{a})}\left(K_0-q_2(\vec{a})\int^t e^{q_{1,\text{reg}}(\tau;\vec{a})}d\tau\right)\right]^{1-\gamma}, \tag{32}$$

where K_0 is a constant of integration. Since $v_0(t;\vec{a})$ will be only piece-wise continuous with jumps at scheduled jump times, K_0 will be different on different intervals between scheduled jumps with separate jump conditions, i.e.,

$$v_0(t;\vec{a})=\left\{\begin{array}{ll} V_{0,\ell}(t;\vec{a}), & T^+_{2,\ell-1}\le t<T^-_{2,\ell},\ \ell=N_2+1\ldots,2,1 \\ V_{0,\ell}(T^-_{2,\ell};\vec{a}), & t=T^-_{2,\ell},\ \ell=N_2,\ldots,2,1 \end{array}\right\}, \tag{33}$$

with semi-open intervals appropriate for right continuous limits and where $T_{2,0}\equiv 0$ and $T_{2,N_2+1}\equiv T$, are the specified starting and stopping times, respectively.

On the final time step, $[T_{2,N_2}, T)$, the optimal utility value function final condition, $v(T,w;\vec{a}) = \mathcal{U}_f(w;\vec{a})$, using (16) and (13), so the partially separated time function satisfies the reduced final condition, $v_0(T;\vec{a}) = 1$. Thus, using (32),

$$v_0(t;\vec{a}) = V_{0,N_2+1}(t;\vec{a}) \equiv \left[e^{-q_{1,\text{reg}}(t;\vec{a})}\left(1 + q_2(\vec{a})\int_t^T e^{q_{1,\text{reg}}(\tau;\vec{a})}d\tau\right)\right]^{1-\gamma}, \tag{34}$$

setting $q_{1,\text{reg}}(T;\vec{a}) \equiv 0$ to fix $q_{1,\text{reg}}$'s constant integration, so that the solution for the optimal value function is $v(t,w;\vec{a}) = \mathcal{U}_f(w;\vec{a})V_{0,N_2+1}(t;\vec{a})$ with optimal controls, $\vec{u}^*(t;\vec{a})$, in absence of control constraints, using solutions from (25), while if there are constraints on $\vec{u}$ such as component-wise constraints, then

$$u_i^*(t;\vec{a}) = \left\{\begin{array}{ll} U_{\min,i}, & u_{\text{reg},i}(t;\vec{a}) \le U_{\min,i} \\ u_{\text{reg},i}(t;\vec{a}), & U_{\min,i} \le u_{\text{reg},i}(t;\vec{a}) \le U_{\max,i} \\ U_{\max,i}, & U_{\max,i} \le u_{\text{reg},i}(t;\vec{a}) \end{array}\right\} \tag{35}$$

for $i = 1, 2, \ldots, N$, and

$$c^*(t,w;\vec{a}) = c_{\text{reg}}(t,w;\vec{a}) = wq_2(\vec{a})/V_{0,N_2+1}^{1/(1-\gamma)}(t;\vec{a}).$$

On earlier time steps $[T_{2,\ell-1}^+, T_{2,\ell}^-)$ between scheduled jumps, for $\ell = N_2, \ldots, 2, 1$, in the natural backward time of dynamic programming, the $v_0(t;\vec{a})$ using (18) and (23) must satisfy the local final jump condition for that interval,

$$V_{0,\ell}(T_{2,\ell}^-;\vec{a}) \quad = \quad I_2(\vec{u}_{2,\ell}^-, T_{2,\ell}^-;\vec{a}), \tag{36}$$

where

$$I_2(\vec{u}, T_{2,\ell}^-;\vec{a}) \equiv \gamma\int_{\mathcal{J}_2}\mathcal{U}(1 + (\vec{J}_{2,\ell}^-)^\top\vec{u})\mathcal{U}_2(1 + J_{2,0,\ell}^-)V_{0,\ell+1}(T_{2,\ell}^+; a_1, (1 + J_{2,0,\ell}^-)a_2)\phi_2(\hat{j}_2)d\hat{j}_2, \tag{37}$$

and the corresponding control is given by the optimal control,

$$\vec{u}_{2,\ell}^-(\vec{a}) \equiv \left[u_i^*(T_{2,\ell}^-;\vec{a})\right]_{N\times 1} = \operatorname*{argmax}_{\vec{u}}\left[I_2(\vec{u}, T_{2,\ell}^-;\vec{a})\right], \tag{38}$$

with the maximization over the constrained control domain, taking the regular control vector when the constraints are satisfied. Also note that since the factor $(1 + \vec{J}_2^\top\vec{u})$ arises as a multiplying factor of wealth in the power utility function, there is in fact another constraint on the combined control $\vec{u}$ and the scheduled jump amplitude $\vec{J}_2$, such that $(1 + \vec{J}_2^\top\vec{u}) \ge 0$ is an additional condition on the selection of $\vec{J}_2$ and $\vec{u}^*$.

With local final condition in (36) and the general solution in (32),

$$v_0(t;\vec{a}) = V_{0,\ell}(t;\vec{a}) \equiv \left[e^{-q_{1,\text{reg}}(t;\vec{a})}\left(e^{q_{1,\text{reg}}(T_{2,\ell};\vec{a})}\left(V_{0,\ell}\right)^{1/(1-\gamma)}(T_{2,\ell}^-;\vec{a}) + q_2(\vec{a})\int_t^T e^{q_{1,\text{reg}}(\tau;\vec{a})}d\tau\right)\right]^{1-\gamma}, \tag{39}$$

for the interval $[T_{2,\ell-1}^+, T_{2,\ell}^-)$ when $\ell = N_2 + 1, \cdots, 2, 1$ subintervals between scheduled jumps, where $V_{0,\ell}(T_{2,\ell}^-;\vec{a})$ is given by (36) with $V_{0,\ell+1}(T_{2,\ell-1}^+;\vec{a}) = V_{0,\ell+1}(T_{2,\ell};\vec{a})$ by piece-wise continuity and right continuous limits supplying the value under the integral on the RHS of (36) with $\ell + 1$ replaced by ℓ. The corresponding optimal consumption is

$$c^*(T_{2,\ell}^-, w;\vec{a}) = c_{\text{reg}}(T_{2,\ell}^-, w;\vec{a}) = wq_2(\vec{a})/\left(V_{0,\ell}\right)^{1/(1-\gamma)}(T_{2,\ell}^-;\vec{a}), \tag{40}$$

which is linear in the wealth, but piece-wise continuous with jumps at the scheduled jump times according to on the jumps of $v_0(t;\vec{a})$. The optimal control on $[T_{2,\ell-1}^+, T_{1,\ell}^-)$ has the same form as in (35) for $[T_{2,N_2}^+, T)$, since it depends only on the diffusive volatility $\sigma(\vec{a})$, the mean appreciation rate $\mu(\vec{a})$ less the interest rate r, the jump amplitudes and their distributions.

6. Further Computational Considerations

In the previous section, the optimal, expected running consumption and terminal wealth investment portfolio problem using power utilities reduces the computational problem to a much more feasible form than that for the more general problem in the Section 4.. The main computational difficulty over the more standard Gaussian noise problem is the numerical treatment of the marked Poisson process related integrals that appear in the reduced equations for the optimal control $\vec{u}^*(t, w; \vec{a})$ in both regular form (25) as well as jump form (38), the growth rate $q_{1,\mathrm{reg}}(t; \vec{a})$ in (29) for the separated time function $v_0(t; \vec{a})$-equation and the jump conditions for $v_0(t; \vec{a})$ in (36).

Hanson and Westman [21] have developed numerical procedures for treating these marked Poisson jump integral that are valid for arbitrary jump densities. The procedure generalizes Gaussian quadrature rules using the density as the integral weighting function unrestricted to normal or exponential distributions. Given a continuous density $\phi(z)$, say $\phi_1(\widehat{j}_1)$, this *Gaussian-Statistics quadrature* for jump integrals approximates the integrals over continuous functions $f(z)$ as

$$\overline{f} \equiv \int_{\mathcal{J}} f(z)\phi(z)dz \simeq \sum_k w_k f(z_k), \tag{41}$$

where the nodes $z_k \in \mathcal{J}$ and corresponding weights w_k are related to the first few moments of the density $\phi(z)$, which for the two point rule has cubic moment accuracy up to and including skewness. Piece-wise rules, with piece-wise renormalization, were also constructed in [21]. In the case where the density is not continuous, such as for discrete distributions, interpolation can be used on the resulting discrete sums at discrete values, $f(z_i)$, for approximations at specified nodes.

The implicit equations governing the regular or optimal controls in (25) and the scheduled jump controls in (38) require some iteration procedure such as Newton's method or can be be solved within the general extrapolator-predictor-corrector procedure summarized by Hanson [6] for computational stochastic dynamic programming problems for Markov noise in continuous-time. Also, the separation imperfection function $\psi(t, \vec{j}_1; \vec{a})$ in (27) in general requires interpolation to evaluate the value function when $(1 + J_{1,0})a_1$ is not an a_1-node. Also, interpolation is a needed for the argument $(1 + J^-_{2,0,\ell})a_2$ in the maximization of the jump integral I_2 in (38).

7. Numerical Test Model

A simple jump model has been formulated for the purposes of code development. It is assumed the state space has the dimension of one stock ($N = 1$) and both unscheduled (random) and scheduled (deterministic) jumps result in two equally likely, discrete random jump amplitude marks at each jump time, i.e.,

$$dS(t) = S(t)\left[\mu(a_1, a_2)dt + \sigma dZ(t) + \int_{\mathcal{J}_1} J_{1,1}(j_1)\mathcal{P}(dt, dj_1) + \sum_{\ell=1}^{N_2} J_{2,1}(j_{2,\ell})dH_R(t - T_{2,\ell})\right], \tag{42}$$

where both the drift $\mu(a_1, a_2)$ and volatility σ are scalar processes. The symbol $H_R(t - T_{2,\ell})$ represents the deterministic step function jump process that is right continuous at jump time $T_{2,\ell}$ and its differential of also a step function in pulse form. The jump amplitudes are assumed to be linear (affine) in the marks, $J_{k,1}(j_{k,1}) = J_{k,1,a} + J_{k,2,b} \cdot j_{k,1}$, for each jump type, $k = 1$ or 2 and are realized through two possible marks $j_{k,1} = 1, 2$, with probabilities $p_{k,1}(j_{k,1}) = 0.5$. For the scheduled process, when $k = 2$, only the jump times are scheduled, but the amplitudes, i.e., the responses, are random.

Since the stock dynamics arise from a geometric jump-diffusion-deterministic process, the logarithmic process by the general chain rule satisfies the SDE,

$$\begin{aligned} d\ln(S(t)) &= [\mu(a_1, a_2) - 0.5\sigma^2]dt + \sigma dZ(t) + \int_{\mathcal{J}_1} \ln(1 + J_{1,1}(j_1))\mathcal{P}(dt, dj_1) \\ &\quad + \sum_{\ell=1}^{N_2} \ln(1 + J_{2,1}(j_{2,\ell}))dH_R(t - T_{2,\ell}), \end{aligned} \tag{43}$$

which has jumps

$$[\ln(S)](T_{k,\ell}) = \ln(1 + J_{k,1}(j_{k,\ell})), \tag{44}$$

for unscheduled jumps when $k = 1$ and scheduled jumps when $k = 2$ with amplitudes determined by the realized marks $j_{k,\ell}$ at jump time $T_{k,\ell}$. The time averaged expectation of the logarithm over the time horizon of one year ($T = 1$) is

$$\mathrm{Aver}[E[\ln(S(t))]] = \mu(a_1, a_2) - 0.5\sigma^2 + 0.5\lambda \sum_{j_1=1}^{2} \ln(1 + J_{1,1}(j_1)) + 0.5\frac{N_2}{T} \sum_{j_2=1}^{2} \ln(1 + J_{2,1}(j_2)) \tag{45}$$

and the time averaged variance is

$$\mathrm{Aver}[\mathrm{Var}[\ln(S(t))]] = \sigma^2 + 0.5\lambda \sum_{j_1=1}^{2} \ln^2(1 + J_{1,1}(j_1)) + 0.5\frac{N_2}{T} \sum_{j_2=1}^{2} \ln^2(1 + J_{2,1}(j_2)). \tag{46}$$

The time averaging is used to smooth out the deterministic jumps in time so the first and second moments can be used to estimate model parameters.

In order to get realistic values for the coefficients, the daily closings of the S&P500 stock index from 1995-1999 [3] are used as a large sample composite estimate of a stock market mutual fund. The S&P500 data has been transformed into changes in the natural logarithm of the index closings from day to day as illustrated in Figure 1. The use of higher order moments for determining the model coefficients are avoided due to the high ill-conditioning in nonlinear curve fitting. The Poisson rate is taken as $\lambda = 3$ per year as a rough estimate of number of extreme outliers in the data corresponding to the day to day changes in the logarithm of the index. Approximate extreme values in the logarithmic changes $dlns(1) = -0.07$ and $dlns(2) = +0.05$ lead to the unscheduled jump linear coefficients $J_{1,1,b} = \exp(dlns(2)) - \exp(dlns(1))$ and $J_{1,1,a} = \exp(dlns(1)) - 1 - J_{1,1,b}$, using jump equation (44). Taking less extreme values for scheduled jumps, $dlns(1) = -0.05$ and $dlns(2) = +0.03$, similarly leads to values for $J_{2,1,b}$ and $J_{2,1,a}$ for the scheduled jump coefficients. With the S&P500 sample standard deviation of 0.01, the volatility σ can be found directly from (46), since all jump process parameters have been specified. The sample time step has been take as $\Delta t = 1/365$ years for simplicity, ignoring weekends and holidays for this simple model. Finally with the volatility determined, the leading drift $\mu(0, 0)$ coefficient follows from the expected drift increment in (45). The parameter processes, (a_1, a_2), are assumed to effect only the drift and in a linear way, so that

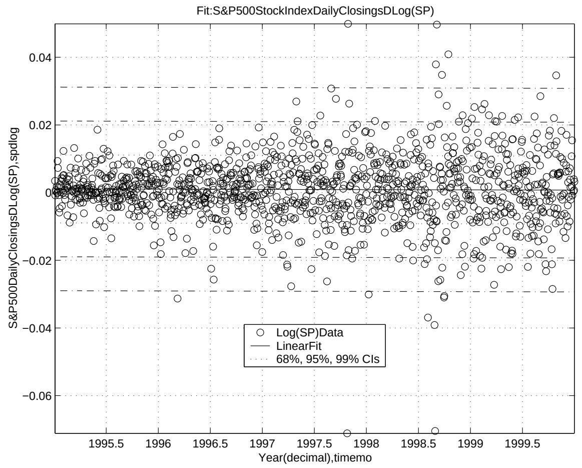

Figure 1: Changes in the logarithm of the S&P500 stock index from one day to the next versus time of the year. Linear fit (light solid line) is nearly zero and horizontal. The confidence intervals for one (68%), two (95%) and three (99%) standard deviations are presented (light dashed lines).

$\mu(a_1, a_1) = \mu(0,0)(1 - 0.1(a_1 + a_2))$. Economic parameters are $r = 0.070537$ from the average rate for Moody AAA bonds and $\beta = 0.046167$ from the average discount rate, both from the Federal Reserve Bank [2] for 1995-1999. The powers of the utility functions were taken as $\gamma = 0.20$ and $\gamma_1 = 0.01 = \gamma_2$ with $N_2 = 12$ scheduled jumps per year in the middle of the month. Other parameters of the parameter processes (a_1, a_2) are taken to be $J_{1,0} = -0.05$ and $J_{2,0} = -0.05$, similar to other jump amplitudes. Control constraints are $U_{\min} = -2.0$ and $U_{\max} = +2.0$, while $C_{\max} = +400.0$ for consumption.

Preliminary results for the scaled value function $v_0(t; \vec{a})$, the optimal expected value $V(t, w; \vec{a})$ when $w = 100.0$, the regular control $u_{\text{reg},1}(t; \vec{a})$ and the optimal control $u_1^*(t; \vec{a})$ are all presented in the subfigures $\{A, B, C, D\}$ of Figure 2, respectively. Here, the parameter process values are set as $\vec{a} = (1, 1)$. The numerical and graphical results were generated using the MATLAB matrix laboratory system Full Version 5.3.1R11.1 [16]. The optimal value appears to be nearly linearly decreasing with time, except for small jump decrements at scheduled jump times. The optimal control also jumps, but in larger size and as jump increments. The optimal consumption has small jump increments at the scheduled jump times while the optimal value jump downward.

8. Conclusions

In this paper, the portfolio optimization model for investment wealth dependent on external jump events introduced by Rishel [17] has been improved and generalized. The underlying stock price, random scheduled and deterministic unscheduled external jump processes have been modeled consistently by Markov noise in continuous time and similarly modeled deterministic processes. The Markov noise includes both Brownian motion and marked Poisson processes, while the deterministic processes are modeled by right continuous generalized functions in the same spirit as the marked Poisson processes leading to a more integrated approach. The expected terminal wealth utility objective of

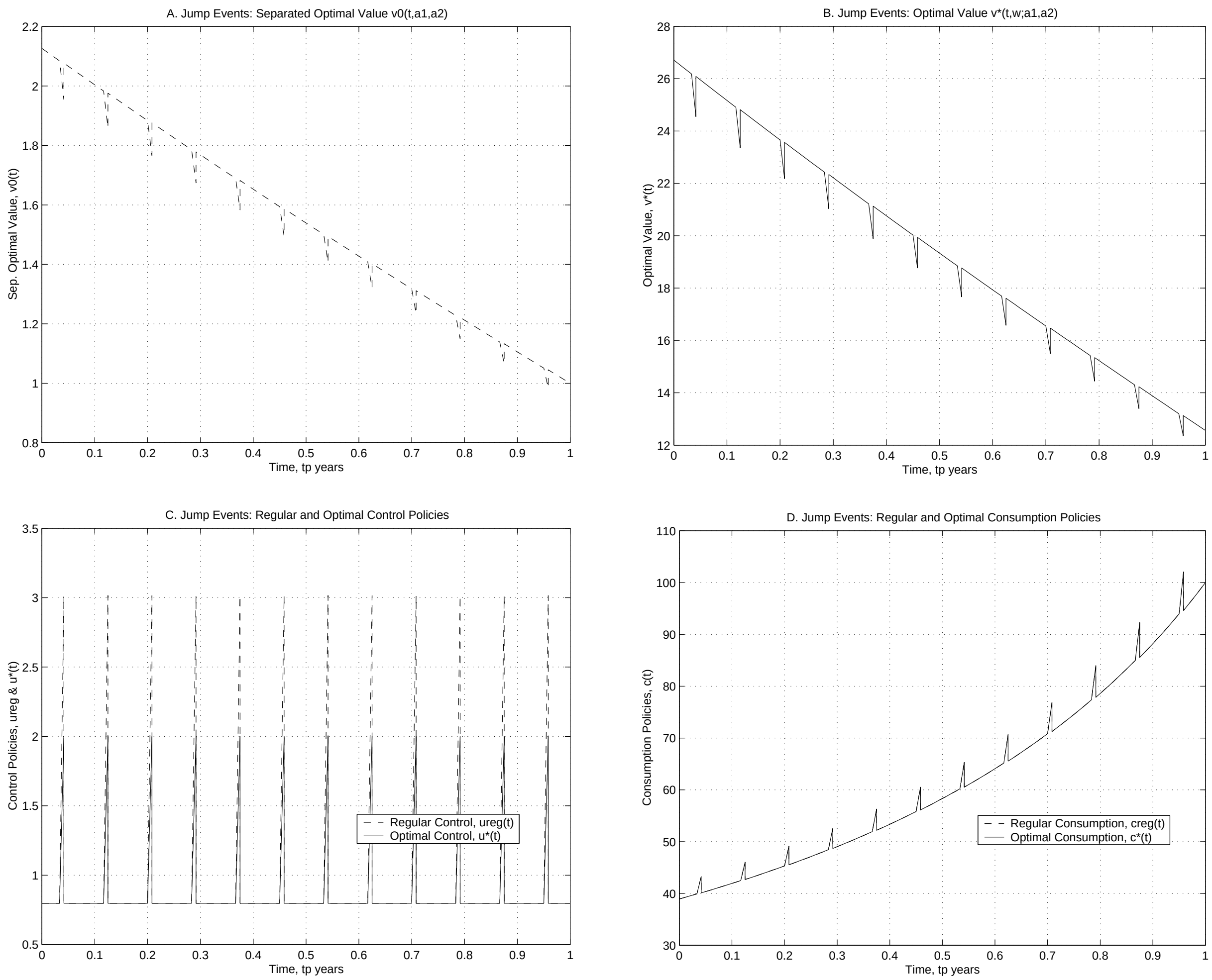

Figure 2: Numerical Results for Test Model. A. Scaled optimal value approximation $v_0(t;1,1)$ (dashed line) versus time t. B. Optimal expected value approximation $V_0^*(t,w;1,1)$ (solid line) versus time t at wealth $w=100$. C. Regular optimal control $u_{\text{reg},1}(t;1,1)$ (dashed line) and constrained optimal control $u_1^*(t;1,1)$ (solid line) approximations versus time t. D. Regular optimal consumption $c_{\text{reg}}(t,100;1,1)$ (dashed line) and $c_1^*(t,100;1,1)$ (solid line) versus time t.

Rishel [17] has been extended by including the scheduled and unscheduled jump parameter in a genuine way through including them in the terminal utility, while consumption has been added in terms of the cumulative instantaneous utility. Discounting has been included in the terminal objective and the instantaneous or running objective as would be in any policy strategy sensitive to other opportunities that might produce a higher gain or interest. Also, constraints are placed on the stock fraction controls to make optimal control computation for the power utility, jump model finite and well-posed. Formulae are carefully worked out for the piece-wise continuous solutions with jump conditions for the power utility models of the constant relative risk aversion type. Overall, the modifications make the optimal portfolio jump model more realistic and computationally feasible. Further computational recommendations are given that include generalized Gaussian quadrature for arbitrary jump amplitude density integrals and predictor-corrector methods for stochastic dynamic programming. Preliminary results are given for a model test problem with two discrete random jump amplitudes, for both scheduled and unscheduled type jump events.

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