

Optimal Production Scheduling for Manufacturing Systems with Preventive Maintenance in an Uncertain Environment

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2001 ACC

September 14, 2000

Abstract

Consider a manufacturing system in which a single consumable good is fabricated in a process that consists of k stages in an uncertain environment. On each stage, there are a number of workstations that are assumed to have different operating parameters that are subject to failure, repair, and preventive maintenance which generate discrete jumps in the value of the state. A Just-In-Time manufacturing discipline is assumed for the workstations with running costs that include penalties for shortfall and surplus production. The formulation presented here for the optimal production scheduling for the manufacturing system requires extensions to the results of the LQGP problem with state dependent Poisson processes (SDPP) by the inclusion of coefficients for the dynamics and the costs that are parameterized by the value of the state. The cost functional used is fully quadratic which is an enhancement for the LQGP problem. The functionality of this canonical model is demonstrated with a numerical example.

1. Introduction

A manufacturing system can generally be categorized as flexible (FMS) or multistage (MMS). The major difference between an FMS and an MMS is the perspective of how the manufacturing system works. An FMS is concerned with local issues such as how individual pieces are routed through the system. Whereas an MMS has the global concern of determining how the production rates for the entire system. An MMS consists of a number of stages at which value is added to each piece. Each stage may be viewed as an FMS. Kimemia and Gershwin [12] describe in detail the differences and similarities between FMS and MMS while providing a hierarchical scheme and algorithm for the operational control of an FMS.

Olsder and Suri [14], proposed a stochastic model that used a homogeneous *jump Markov processes* to model the transitions of workstations between the failed and operational states. They stated the usefulness of the model is limited by the ability to solve the Hamilton-Jacobi-Bellman (HJB) partial differential equation of dynamic programming, due

* Work supported in part by the National Science Foundation Grant DMS-99-73231.

† Work supported in part by the Natural Sciences and Engineering Research Council of Canada under grants OGP0036444

to the exponential growth of computational and memory resources needed to solve the HJB equation employing finite differences, commonly referred to as the *Curse of Dimensionality* [2].

Boukas and Haurie [4] present a model for a continuous-time stochastic flow control for production scheduling with preventive maintenance. In this treatment, the failure and preventive maintenance rates depend on the operational age of the workstation, which is defined as the time since the last repair or preventive maintenance. A control variable is used to determine when to perform preventive maintenance in order to avoid failures in an optimal way. An irreducible Markov chain is used to generate the transitions between the operational, failed, and preventive maintenance states for the workstations, which consists of 3^N states for N workstations. A numerical method based on Kushner's Markov Chain Approximation [13] is used to approximate the solution for the dynamic programming problem, which requires a discrete mesh. In this method, the mesh selected is finite and artificial boundary conditions are used which introduces additional error. In the example presented for 2 workstations and 1 part type, using 21 points for the mesh of the production surplus and operational ages for the workstations, the total number of mesh points is $21^3 * 3^2 = 83,349$. If another workstation is included the number of mesh points is $21^4 * 3^3 = 5,250,987$, clearly demonstrating the *Curse of Dimensionality*, and therefore would not be suitable for a large number of workstations.

In this paper we extend the production scheduling model of Westman and Hanson [22] and modify the Linear Quadratic Gaussian Poisson (LQGP) problem [16] with state dependent Poisson processes (SDPP) [19] to model and solve the optimal control problem. The manufacturing system is a hybrid model for production scheduling consisting of local concerns in managing the status of workstations and global concerns for meeting the demand. The manufacturing system consists of k stages. The finished product of stage k is the base input for stage $k + 1$. On each stage there are a number of workstations that are assumed to have different operating parameters and are subject to failure and repair. Each workstation can undergo preventive maintenance to reduce the operational age to a lower level so that failures rarely occur. It is assumed that the preventive maintenance is beneficial in that its cost is much less than repair and takes less time to perform. In [22], pieces are left in buffers between stages so that the plant manager must manually adjust the production rates to consume them, this is not the case here.

In this paper, the LQGP problem [16] with state dependent Poisson processes (SDPP) [19] is revised to include a full quadratic form for the cost functional. The coefficients for the cost and the dynamics are parameterized by the value of the state. This formulation allows for a greater ability to model physical reality in the dynamics and cost constraints in a simple way. The formal solution of the LQGP problem is derived which consists of a system of unidirectional coupled nonlinear ordinary differential equations. The LQGP problem does not suffer from the *Curse of Dimensionality* and requires $O(n^2/2)$ elements in the asymptotic limit for large n , the dimension of the state space. This formulation is applied to production scheduling of a manufacturing system is used to demonstrate the use of these refinements.

The paper is arranged as follows. In Section 2., the LQGP Problem with state dependent Poisson noise is extended to accommodate state parameterized coefficients and a full quadratic cost functional well as numerical scheme to solve the resulting control problem. In Section 3., a LQGP problem [16] utilizing state dependent Poisson processes [19] is used to formulate the dynamical system for the manufacturing system and in Section 4., a numerical example is

presented.

2. LQGP Problem Expansion

The canonical form for the LQGP problem used here appears in Westman and Hanson [16], for the case with state independent Poisson noise, and in [19] for state dependent Poisson processes. The LQGP problem is characterized by *linear* deterministic dynamics, *quadratic* costs, *Gaussian* noise disturbance, and *Poisson* jumps in the state value. In this treatment the LQGP problem is expanded with the dynamic and cost coefficients to be parameterized by the value of the state, i.e., a *quasi-LQGP* problem. This allows for greater flexibility of modeling in a convenient notational form. Additionally, the cost functional used is the full quadratic form which extends the classic LQGP cost functional. Considerations for modeling physical systems are presented, as well as formal solution to the LQGP problem.

The quasi-linear dynamical system for the quasi-LQGP problem is governed by the stochastic differential equation (SDE) subject to Gaussian and state dependent Poisson processes disturbances is given by

$$\begin{aligned} d\mathbf{X}(t) &= [A(t; \mathbf{X}(t))\mathbf{X}(t) + B(t; \mathbf{X}(t))\mathbf{U}(t) + \mathbf{C}(t; \mathbf{X}(t))]dt + G(t; \mathbf{X}(t))d\mathbf{W}(t) \\ &+ [H_1(t; \mathbf{X}(t))\mathbf{X}(t)]d\mathbf{P}_1(\mathbf{X}(t), t) + [H_2(t; \mathbf{X}(t))\mathbf{U}(t)]d\mathbf{P}_2(\mathbf{X}(t), t) + H_3(t; \mathbf{X}(t))d\mathbf{P}_3(\mathbf{X}(t), t), \end{aligned} \quad (1)$$

for general Markov processes in continuous time, with $m \times 1$ state vector $\mathbf{X}(t)$, $n \times 1$ control vector $\mathbf{U}(t)$, $r \times 1$ Gaussian noise vector $d\mathbf{W}(t)$, and $q_\ell \times 1$ space-time Poisson noise vectors $d\mathbf{P}_\ell(\mathbf{X}(t), t)$, for $\ell = 1$ to 3 . Note that the term $[H_1(t; \mathbf{X}(t)) \cdot \mathbf{X}(t)]d\mathbf{P}_1(\mathbf{X}(t), t)$ is not linear in the state. The dimensions of the respective coefficient matrices are: $A(t; \mathbf{x})$ is $m \times m$, $B(t; \mathbf{x})$ is $m \times n$, $\mathbf{C}(t; \mathbf{x})$ is $m \times 1$, $G(t; \mathbf{x})$ is $m \times r$, while the $H_\ell(t; \mathbf{x})$ are dimensioned, so that $[H_1(t; \mathbf{x}) \cdot \mathbf{x}] = [\sum_k H_{1ijk}(t; \mathbf{x})x_k]_{m \times q_1}$, $[H_2(t; \mathbf{x}) \cdot \mathbf{u}] = [\sum_k H_{2ijk}(t; \mathbf{x})u_k]_{m \times q_2}$ and $H_3(t) = [H_{3ij}(t; \mathbf{x})]_{m \times q_3}$. The coefficients for the dynamical system can depend on the value of the state as a parameter, and numerically must be evaluated first, i.e., we assume the coefficients are subdominant or are locally in the state. Note that the space-time Poisson terms are formulated to maintain the linear nature of the dynamics, but the first two are actually bilinear in either $\mathbf{X}(t)$ or $\mathbf{U}(t)$ with $d\mathbf{P}_\ell$ for $\ell = 1$ or 2 , respectively.

The *state dependent Poisson process* can be viewed as a sequence of events that is represented by its i th couple $\{T_i(\mathbf{X}(T_i)), M_i(\mathbf{X}(T_i))\}$, for $i = 1$ to k , where $T_i(\mathbf{X}(T_i))$ is the time for the occurrence of the i th jump with state dependent mark amplitude $M_i(\mathbf{X}(T_i))$. The form for the SDPP allows for a great deal of realism to be included in the model for deterministic and random jumps in the evolution of the state values. A wider range of stochastic control applications can be accurately modeled since the arrival rates and amplitudes can depend on the value of the state. Additionally, this formulation allows for simpler dynamical system modeling of complex random phenomena. The inclusion of state dependence in the Poisson noise means that the quasi-LQGP dynamics (1) is nonlinear in the state vector.

The state dependent vector valued marked Poisson noises are related to the Poisson random measure (see Gihman and Skorohod [8] or Hanson [10]) and are defined as

$$d\mathbf{P}_\ell(\mathbf{X}(t), t) = [dP_{\ell,i}(\mathbf{X}(t), t)]_{q_\ell \times 1} = \left[\int_{\mathcal{Z}_{\ell,i}} z_i \mathcal{P}_{\ell,i}(dz_i, \mathbf{X}(t), dt) \right]_{q_\ell \times 1}, \quad (2)$$

for $\ell = 1$ to 3 which consists of q_ℓ independent differentials of space-time Poisson processes that are functions of the state, $\mathbf{X}(t)$, where z_i is the Poisson jump amplitude random variable or the mark of the $dP_{\ell,i}(\mathbf{X}(t), t)$ Poisson process where $\ell = 1$ to 3 and $i = 1$ to q_ℓ . The mean or expectation is given by

$$\text{Mean}[d\mathbf{P}_\ell(\mathbf{X}(t), t)] = \Lambda_\ell(\mathbf{X}(t), t)dt \int_{\mathcal{Z}_\ell} \mathbf{z}\phi_\ell(\mathbf{z}, \mathbf{X}(t), t)d\mathbf{z} \equiv \Lambda_\ell(\mathbf{X}(t), t)\bar{\mathbf{Z}}_\ell(\mathbf{X}(t), t)dt, \quad (3)$$

where $\Lambda_\ell(\mathbf{X}(t), t)$ is the diagonal matrix representation of the state dependent Poisson arrival rates $\lambda_{\ell,i}(\mathbf{X}(t), t)$ for $\ell = 1$ to 3 and $i = 1$ to q_ℓ , $\bar{\mathbf{Z}}_\ell(\mathbf{X}(t), t)$ is the mean of the jump amplitude mark vector and $\phi_{\ell,i}(z_i, \mathbf{X}(t), t)$ is the density of the (ℓ, i) th amplitude mark component. Assuming component-wise independence, $d\mathbf{P}_\ell(\mathbf{X}(t), t)$ has covariance given by

$$\text{Covar}[d\mathbf{P}_\ell(\mathbf{X}(t), t), d\mathbf{P}_\ell^\top(\mathbf{X}(t), t)] = \Lambda_\ell(*)dt \int_{\mathcal{Z}_\ell} (\mathbf{z} - \bar{\mathbf{Z}}_\ell(*))(\mathbf{z} - \bar{\mathbf{Z}}_\ell(*))^\top \phi_\ell(\mathbf{z}, *)d\mathbf{z} \equiv \Lambda_\ell(*)\sigma_\ell(*)dt, \quad (4)$$

with, for instance, $\sigma_\ell(*) = \sigma_\ell(\mathbf{X}(t), t) = [\sigma_{\ell,i,j}\delta_{i,j}]_{q_\ell \times q_\ell}$ denoting the diagonalized covariance of the amplitude mark distribution for $d\mathbf{P}_\ell(\mathbf{X}(t), t)$. Again, the mark vector is not assumed to have a zero mean, i.e., $\bar{\mathbf{Z}}_\ell \neq 0$, permitting additional modeling complexity. Note, that for discrete distributions the above integrals need to be replaced by the appropriate sums. The Gaussian white noise term, $d\mathbf{W}(t)$, consists of r independent, standard Wiener processes $dW_i(t)$, for $i = 1$ to r . These Gaussian components have zero infinitesimal mean, $\text{Mean}[d\mathbf{W}(t)] = \mathbf{0}_{r \times 1}$ and diagonal covariance. $\text{Covar}[d\mathbf{W}(t), d\mathbf{W}^\top(t)] = I_r dt$. It is further assumed that all of the individual component terms of the Gaussian noise are independent of all of the Poisson processes.

The j th jump of the $\{\ell, i\}$ th space-time Poisson process at time $t_{\ell,i,j}$ with mark amplitude $z_i = M_{\ell,i,j}$ causes the following jump from $t_{\ell,i,j}^-$ to $t_{\ell,i,j}^+$ in the state:

$$[\mathbf{X}](t_{\ell,i,j}) = \left\{ \begin{array}{ll} [H_1(t_{\ell,i,j}; \mathbf{X}(t_{\ell,i,j}^-))\mathbf{X}(t_{\ell,i,j}^-)]_i M_{\ell,i,j}, & \text{if } \ell = 1 \\ [H_2(t_{\ell,i,j}; \mathbf{X}(t_{\ell,i,j}^-))\mathbf{U}(t_{\ell,i,j}^-)]_i M_{\ell,i,j}, & \text{if } \ell = 2 \\ [H_3(t_{\ell,i,j}; \mathbf{X}(t_{\ell,i,j}^-))]_i M_{\ell,i,j}, & \text{if } \ell = 3 \end{array} \right\}. \quad (5)$$

The cost functional or performance index that is used is given by the *time-to-go* or *cost-to-go* functional form:

$$V[\mathbf{X}, \mathbf{U}, t] = \frac{1}{2}\mathbf{X}^\top(t_f)S(t_f)\mathbf{X}(t_f) + \int_t^{t_f} C(\mathbf{X}(\tau), \mathbf{U}(\tau), \tau)d\tau, \quad (6)$$

where the time horizon is (t, t_f) , with $S(t_f) \equiv S_f$ is the quadratic final cost coefficient matrix and $C(\mathbf{x}, \mathbf{u}, t)$ is quadratic running cost function. The final cost, known as the *salvage cost*, is given by the quadratic form, $\mathbf{x}^\top S_f \mathbf{x} = S_f : \mathbf{x}\mathbf{x}^\top = \text{Trace}[S_f \mathbf{x}\mathbf{x}^\top]$, where the double dot product is defined by $A : B = \sum_i \sum_j A_{i,j} B_{i,j} = \text{Trace}[AB^\top]$. The form for the instantaneous cost employed here is the general, quasi-quadratic form:

$$C(\mathbf{x}, \mathbf{u}, t) = \frac{1}{2}[\mathbf{x}^\top Q_2(t; \mathbf{x})\mathbf{x} + \mathbf{u}^\top R_2(t; \mathbf{x})\mathbf{u} + \mathbf{x}^\top C_2(t; \mathbf{x})\mathbf{u}] + Q_1^\top(t; \mathbf{x})\mathbf{x} + R_1^\top(t; \mathbf{x})\mathbf{u} + C_0(t; \mathbf{x}) \quad (7)$$

where the coefficient matrices are of the appropriate dimension. This form allows for a more robust form for the performance index since the coefficients are parameterized by and assumed to be weakly dependent on the state in a general quasi-quadratic form. In order to minimize (6) with instantaneous cost function (7) requires that the quadratic control cost coefficient $R_2(t; \mathbf{x})$ is assumed to be a symmetric positive definite $n \times n$ array, while the quadratic state control coefficient $Q_2(t; \mathbf{x})$ is assumed to be a symmetric positive semi-definite $m \times m$ array. The LQGP problem is defined by (1, 6, 7).

For the stochastic dynamic programming formulation, the *optimal, expected cost*, is taken to be

$$v(\mathbf{x}, t) \equiv \text{Min}_{\mathbf{u}[t, t_f]} \left[\text{Mean}_{\mathbf{P}, \mathbf{W}[t, t_f]} [V[\mathbf{X}, \mathbf{U}, t] \mid \mathbf{X}(t) = \mathbf{x}, \mathbf{U}(t) = \mathbf{u}] \right], \quad (8)$$

where the restrictions on the state and control are that they belong to the admissible classes for the state, $\mathcal{D}_{\mathbf{x}}$, and control, $\mathcal{D}_{\mathbf{u}}$, respectively. A final condition on the optimal, expected value, $v(\mathbf{x}, t_f) = \frac{1}{2} \mathbf{x}^\top S_f \mathbf{x}$, for $\mathbf{x} \in \mathcal{D}_{\mathbf{x}}$, is determined from the final cost using (8) with $V[\mathbf{X}, \mathbf{U}, t_f]$ in (6,7) is given by

$$v(\mathbf{x}, t_f) = \frac{1}{2} \mathbf{x}^\top S_f \mathbf{x}, \text{ for } \mathbf{x} \in \mathcal{D}_{\mathbf{x}}. \quad (9)$$

Upon applying the principle of optimality to the optimal, expected performance index, (8, 6, 7) and the chain rule for Markov stochastic processes in continuous time for the LQGP problem yields

$$\begin{aligned} 0 &= \frac{\partial v}{\partial t}(\mathbf{x}, t) + \text{Min}_{\mathbf{u}} \left[(A(t; \mathbf{x})\mathbf{x} + B(t; \mathbf{x})\mathbf{u} + \mathbf{C}(t; \mathbf{x}))^\top \nabla_{\mathbf{x}} [v](\mathbf{x}, t) \right. \\ &+ \frac{1}{2} (GG^\top)(t; \mathbf{x}) : \nabla_{\mathbf{x}} [\nabla_{\mathbf{x}}^\top [v]](\mathbf{x}, t) + C(\mathbf{x}, \mathbf{u}, t) \\ &+ \sum_{k=1}^{q_1} \lambda_{1,k}(\mathbf{x}, t) \int_{\mathcal{Z}_{1,k}} [v(\mathbf{x} + [H_1(t; \mathbf{x}) \cdot \mathbf{x}]_k z_k, t) - v(\mathbf{x}, t)] \phi_{1,k}(z_k, \mathbf{x}, t) dz_k \\ &+ \sum_{k=1}^{q_2} \lambda_{2,k}(\mathbf{x}, t) \int_{\mathcal{Z}_{2,k}} [v(\mathbf{x} + [H_2(t; \mathbf{x}) \cdot \mathbf{u}]_k z_k, t) - v(\mathbf{x}, t)] \phi_{2,k}(z_k, \mathbf{x}, t) dz_k \\ &\left. + \sum_{k=1}^{q_3} \lambda_{3,k}(\mathbf{x}, t) \int_{\mathcal{Z}_{3,k}} [v(\mathbf{x} + \mathbf{H}_{3,k}(t; \mathbf{x}) z_k, t) - v(\mathbf{x}, t)] \phi_{3,k}(z_k, \mathbf{x}, t) dz_k \right]. \quad (10) \end{aligned}$$

The backward partial differential equation (PDE) (10) is known as the Hamilton-Jacobi-Bellman (HJB) equation. The argument of the minimum is the optimal control, $\mathbf{u}^*(\mathbf{x}, t)$, with the regular control, $\mathbf{u}_{\text{reg}}(\mathbf{x}, t)$, defined as the optimal control prior to the application of the control constraints. To solve (10) subject to the final condition, (9), for the LQGP problem a modification of the formal state decomposition of the solution for the usual LQG problem (for the usual LQG, see Bryson and Ho [6]) is assumed:

$$v(\mathbf{x}, t) = \frac{1}{2} \mathbf{x}^\top S(t) \mathbf{x} + \mathbf{D}^\top(t) \mathbf{x} + E(t) + \frac{1}{2} \int_t^{t_f} (GG^\top)(\tau) : S(\tau) d\tau. \quad (11)$$

The terminal condition (9) is satisfied, provided that $S(t_f) = S_f$, $\mathbf{D}(t_f) = \mathbf{0}$, and $E(t_f) = 0$.

The ansatz (11) would not, in general, be true for the state dependent case, but would be applicable if the Poisson processes, dynamic and cost coefficients are locally state independent, while globally state dependent. That is, the state domain is decomposed into subdomains, $\mathcal{D}_{\mathbf{x}} = \bigcup_i \mathcal{D}_{x_i}$, where the arrival rates and moments for all the Poisson processes are constant and the coefficients for the dynamics and costs are independent of the state in the subdomains \mathcal{D}_{x_i} . If there are any explicit dependence on $\mathbf{X}(t)$ then the resulting system would then form a LQGP/U problem, i.e., in LQGP in control only (for more details see Westman and Hanson [17, 18, 19, 20]). Equation (11) would be only approximately valid in the case of subdominant (small) dependence of the coefficients on the state.

Assuming the ansatz (11), using domain decomposition if needed, holds the regular, unconstrained optimal control, $\mathbf{u}^* = \mathbf{u}_{\text{reg}}$, for the region \mathcal{D}_{x_i} dropping the subscripts i is given by

$$\mathbf{u}_{\text{reg}}(t) = -\widehat{R}_2^{-1}(t) \left[\widehat{B}(t) \mathbf{x} + \widehat{D}(t) \right], \quad (12)$$

where $\widehat{R}_2(t)$, given below, is related to $R_2(t)$ but with Poisson term corrections. For the region \mathcal{D}_{x_i} , dropping the

subscripts i and assuming regular control, the coefficients for the optimal expected performance (11) are given by

$$0_{m \times m} = \dot{S}(t) + \left[A^\top S + SA + Q_2 + \tilde{\Gamma}_1 - \hat{B}^\top \hat{R}_2^{-1} \hat{B} \right] (t), \quad (13)$$

$$0_{m \times 1} = \dot{\mathbf{D}}(t) + \left[(A + H_1^\top \Lambda_1 \bar{\mathbf{Z}}_1)^\top \mathbf{D} + \mathbf{Q}_1 + S(\mathbf{C} + H_3 \Lambda_3 \bar{\mathbf{Z}}_3) - \hat{B}^\top \hat{R}_2^{-1} \hat{\mathbf{D}} \right] (t), \quad (14)$$

$$0 = \dot{E}(t) + \left[(\mathbf{C} + H_3 \Lambda_3 \bar{\mathbf{Z}}_3)^\top \mathbf{D} + C_0 + \frac{1}{2} \tilde{\Gamma}_3 - \frac{1}{2} \hat{\mathbf{D}}^\top \hat{R}_2^{-1} \hat{\mathbf{D}} \right] (t), \quad (15)$$

where

$$\Gamma_1(t) \equiv \left[([H_1^\top]_i S [H_1]_j) : (\Lambda_1 \bar{\mathbf{Z}} \bar{\mathbf{Z}}_1) (t) \right]_{m \times m} + 2 \left[(\Lambda_1 \bar{\mathbf{Z}}_1)^\top H_1^\top S \right] (t),$$

$$\Gamma_2(t) \equiv \left[([H_2^\top]_i S [H_2]_j) : (\Lambda_2 \bar{\mathbf{Z}} \bar{\mathbf{Z}}_2) (t) \right]_{n \times n}, \quad \Gamma_3(t) \equiv \left[(H_3^\top S H_3) : (\Lambda_3 \bar{\mathbf{Z}} \bar{\mathbf{Z}}_3) \right] (t), \quad \tilde{\Gamma}_\ell(t) \equiv (\Gamma_\ell + \Gamma_\ell^\top) (t)$$

$$\bar{\mathbf{Z}} \bar{\mathbf{Z}}_\ell(t) \equiv \bar{\sigma}_\ell(t) + \left(\bar{\mathbf{Z}}_\ell \bar{\mathbf{Z}}_\ell^\top \right) (t) = [\sigma_{\ell,i} \delta_{i,j} + \bar{Z}_{\ell,i} \bar{Z}_{\ell,j}]_{q_\ell \times q_\ell}, \quad \hat{R}_2(t) \equiv R_2(t) + \tilde{\Gamma}_2(t),$$

$$\hat{B}(t) \equiv \left[(B(t) + H_2 \Lambda_2 \bar{\mathbf{Z}}_2)^\top S + \frac{1}{2} C_2^\top \right] (t), \quad \hat{\mathbf{D}}(t) \equiv \left[(B(t) + H_2 \Lambda_2 \bar{\mathbf{Z}}_2)^\top \mathbf{D} + \mathbf{R}_1 \right] (t),$$

for $\ell = 1$ to 3. Since the matrix R_2 is positive definite, R_2^{-1} exists and then so does \hat{R}_2^{-1} . Note (13) appears to have Riccati-like quadratic form, but in general is highly nonlinear due to the $S(t)$ dependence on $\hat{R}_2(t)$ through $\Gamma_2(t)$. If $H_\ell = [H_{\ell,i,j,k}]_{m \times q_\ell \times m_\ell}$, then $H_\ell^\top = [H_{\ell,j,i,k}]_{q_\ell \times m \times m_\ell}$.

Due to uni-directional coupling of these matrix differential equations, it is assumed that the nonlinear matrix differential equation (13) for $S(t)$ is solved first and the result for $S(t)$ is substituted into equation (14) for $\mathbf{D}(t)$, which is then solved, and then both results for $S(t)$ and $\mathbf{D}(t)$ are substituted into equation (15) for the state-control independent term $E(t)$. Since the quadratic form in (11) depends only on the symmetric part of $S(t)$, only a triangle part of $S(t)$ need be solved, or $n \cdot (n + 1)/2$ component equations. Thus, for the whole coefficient set $\{S(t), \mathbf{D}(t), E(t)\}$, only $n \cdot (n + 1)/2 + n + 1$ component equations need to be solve, so that for large n the count is $\mathcal{O}(n^2/2)$, asymptotically, which is the same order of effort in getting the triangular part of $S(t)$.

3. Production Scheduling Manufacturing System Model

Consider a manufacturing process that requires k sequential steps to fabricate a single consumable commodity. The planning horizon for the production run is $[0, t_f]$ with a demand of $d(t)$ pieces per unit time. In this formulation, the *loading* and *unloading* stages, the means by which raw materials are introduced and finished goods exit the manufacturing system, respectively, are not considered. The model utilizes FMS-like considerations for the workstations while employing MMS-like criteria for determining the optimal production rates for the each workstation on all stages.

This treatment extends the work of Westman and Hanson [22] in the formulation for the model using coefficients parameterized by the value of the state. One drawback of [22] is that the pieces can be left in the buffers and that it is left up to the plant manager to decide how to consume these pieces which is not the case here. Additionally, a uniform penalty was imposed for not meeting the desired production goal in terms of shortfalls and surplus production. Here a different discipline is used for the *Just In Time* [9] production employing different penalties for surplus and shortfalls in production for all time $t \in [0, t_f]$. The SDPP used in this treatment that describe the changes in the status of the workstations are modeled using coefficients that are parameterized by the state, this is also the case for the cost functional.

3.1. Local Workstation State Equations

For stage k , assume there are N_k workstations that have different operational parameters. Therefore, the status for each workstation is a set of state variables that are related. The events for the workstations are repair, failure, and preventive maintenance, which are mutually exclusive. The tracking of these events along with the operational age of the workstation are the state variables for a given workstation on a given stage. This leads to a high dimensional state space, however since the model is in the form of the expanded LQGP problem presented in Section 2. and does not suffer from the *Curse of Dimensionality*. The state variables for workstation i on stage k are the available production capacity, $r_{ki}(t)$, the operational status (repair/failure), $o_{ki}(t)$, the preventive maintenance status, $m_{ki}(t)$, and the operational age, $a_{ki}(t)$, and is denoted by

$$\mathbf{x}_{ki}(t) = [r_{ki}(t), o_{ki}(t), m_{ki}(t), a_{ki}(t)]^T. \quad (16)$$

The production for a given stage is distributed across all of the workstations evenly. A parameter for each stage k is the production rate, $c_k(t)$, which represents the utilization or the fraction of time busy. The goal of the optimal control problem for the production scheduling is to determine the production rates for each stage for the planning horizon. The production rates need to compensate for changes in the status of the workstations while maintaining the desired constraints on meeting the production demand in the specified way.

The arrival rate, mean time till an event occurs, for failures is dependent on the operational age of the workstation. This implies that the probability of a failure is an increasing function of the operational age of the workstation. Therefore, preventive maintenance is performed periodically based on the operational age of the workstation, which reduces the operational age of the workstation and thereby reduces the probability for a failure to occur. It is assumed that the election to perform preventive maintenance is rational, that is the amount of time perform preventive maintenance is much less than that of repairing a failed machine and/or the cost for preventive maintenance is much less than that of repair. In this treatment, preventive maintenance reduces the available production capacity for the workstation by a fixed percentage assumed to be greater than 0.

The operational and preventive maintenance status values evolve according to stochastic differential equations that are modeled as SDPP with coefficients that are parameterized by the value of the state. These coefficients partition the state space into regions in which events can occur based on the value of the state. For example, a workstation must be operational for a failure to occur, thereby disabling the repair process. This formulation ensures that events can only occur when allowable, thus there are no problems with the stochastic processes around boundaries. The status values are in the range $[0, 1]$ which represents the maximum percentage of available production capacity.

At time t , a given workstation can be in one of the following states operational, under repair, or in maintenance. These events are mutually exclusive, and transitions between under repair and in maintenance are not allowed. The available production capacity is an indicator that determines that the state of a workstation. The available production capacity for workstation i on stage k should be given by $r_{ki}(t) = \text{Min}[o_{ki}(t), m_{ki}(t)]$, but unfortunately this nonlinear expression is not allowable under the LQGP problem. Instead, using the mutually exclusivity of the events, we

introduce another variable for the production state of each workstation defined by

$$dr_{ki}(t) = do_{ki}(t) + dm_{ki}(t). \quad (17)$$

Let M_{ki} represent the maximum number of pieces that can be produced per unit time on workstation i , then the total number of pieces that can be produced on stage k at time t is $\widehat{M}_k(t) = \sum_{i=1}^{N_k} M_{ki} r_{ki}(t) \equiv \mathbf{M}_k^\top \mathbf{r}_k(t)$.

The mean time between failures and the time for repair are assumed to be exponentially distributed and dependent on the operational age of the workstation. The form of the defining equation for the operational status of the workstations is modeled from (1) as the following term:

$$do_{ki}(t) = H_3(t; \mathbf{x}_{ki}(t)) d\mathbf{P}_3(\mathbf{x}_{ki}(t), t) = \begin{bmatrix} \delta_{o_{ki}(t),0} & 0 \\ 0 & -\delta_{r_{ki}(t),1} \end{bmatrix} \begin{bmatrix} d\mathbf{P}_{ki}^R(\mathbf{x}_{ki}(t), t) \\ d\mathbf{P}_{ki}^F(\mathbf{x}_{ki}(t), t) \end{bmatrix}, \quad (18)$$

where the superscripts R and F denote repair and failure processes, respectively. The coefficient matrix, $H_3(t; \mathbf{X}(t))$, is parameterized by the state so that only allowable events may occur which partitions the state space into regions. The $d\mathbf{P}_3(\mathbf{X}(t), t)$ is the SDPP providing the jumps for workstation repair and failure processes with the following properties, $1/\lambda^R(\mathbf{x}_{ki}(t), t) = T_{ki}^R$, $1/\lambda^F(\mathbf{x}_{ki}(t), t) = T_{ki}^F - a_{ki}(t)$, $\overline{Z}_{ki}^R = \overline{Z}_{ki}^F = 1$ and $\sigma_{ki}^R = \sigma_{ki}^F = 0$, where T_{ki}^R and T_{ki}^F are the mean times between repair and failure, respectively. The operational status formulated here can either have a value of 1 which denotes an operational workstation or 0 which denotes a failed workstation not capable of production.

A similar formulation to the operational status is used for the preventive maintenance status. Preventive maintenance is performed in a deterministic fashion that depends on the operational age of the workstation. The effects of maintenance are considered only if the maintenance will be performed in the remaining production horizon. It is assumed that preventive maintenance reduces the amount of available production capacity, but does not necessarily disable production. The SDPPs are used to generate the jumps in the state for the events of beginning maintenance (denoted with a superscript of M) and for the completion of maintenance (denoted with a superscript of D). The defining equation is given by

$$dm_{ki}(t) = \begin{bmatrix} 1 - \delta_{r_{ki}(t),1} & 0 \\ 0 & -c_{ki}^M(t; a_{ki}(t)) \delta_{r_{ki}(t),1} \end{bmatrix} \begin{bmatrix} d\mathbf{P}_{ki}^D(\mathbf{x}_{ki}(t), t) \\ d\mathbf{P}_{ki}^M(\mathbf{x}_{ki}(t), t) \end{bmatrix}, \quad (19)$$

where

$$c_{ki}^M(t; a_{ki}(t)) = \begin{cases} 1, & \text{if } T_{ki}^M - a_{ki}(t) < ttg(t) \\ 0, & \text{if } T_{ki}^M - a_{ki}(t) \geq ttg(t) \end{cases} \quad (20)$$

is used to ensure that maintenance occurs within the production horizon with the time-to-go for the production horizon given by $ttg(t) = T - t$. The sojourn times for these processes are given by $1/\lambda^D(\mathbf{x}_{ki}(t), t) = T_{ki}^D$ and $1/\lambda^M(\mathbf{x}_{ki}(t), t) = T_{ki}^M - a_{ki}(t)$, where T_{ki}^D and T_{ki}^M are the average duration of the maintenance and the mean time between preventive maintenance, respectively. It is assumed that preventive maintenance should be performed before a failure which implies $T_{ki}^M < T_{ki}^F$. The moments for the SDPP for preventive maintenance should be modeled as the average loss of production capacity, \overline{Z}_{ki}^M , and the variance of the loss of production capacity σ_{ki}^M . The duration of preventive maintenance process (D) is used to restore the value for the preventive maintenance status to 1 and has moments given by $\overline{Z}_{ki}^D = \overline{Z}_{ki}^M$ and $\sigma_{ki}^D = 0$.

The current operational age of the workstation is a monotone increasing function of time and of the number of pieces produced which can be reset to a lower level by the performance of repair or preventive maintenance. The age of the workstation evolves according to:

$$da_{ki}(t) = f(c_k(t), t)dt - H_{ki}^D(t; \mathbf{x}_{ki}(t))dP_{ki}^D(\mathbf{x}_{ki}(t), t) - H_{ki}^R(t; \mathbf{x}_{ki}(t))dP_{ki}^R(\mathbf{x}_{ki}(t), t), \quad (21)$$

where $H_{ki}^D(t; \mathbf{x}_{ki}(t))$ and $H_{ki}^R(t; \mathbf{x}_{ki}(t))$ are the coefficients that are used to reset the operational age due to maintenance and repair to a specified lower level, respectively, with $a_{ki}(\tau_{ki}) = \bar{a}(\tau_{ki})$ where τ_{ki} is the time of the last reset.

3.2. Global Surplus State Equation

The global surplus state equation is used to track the production of each stage k to that of the demand expressed as the number of parts per unit time. The corresponding global state variable $s_k(t)$, the surplus aggregate level, represents the surplus if positive or shortfall if negative of the production that have successfully completed k stages. The control, $u_k(t)$, expressed as the number of parts per unit time, is used to adjust the production rates $c_k(t)$ to compensate for changes in workstation status and small local effects modeled as Gaussian process. The ideal for the manufacturing system is to have $s_k(t) = 0$ for all k and t which can be done in the unconstrained case. However, the resulting production rates may not be physically realizable, thus resulting in a surplus or shortfall.

The state equation for the surplus aggregate level for stage k is given by

$$ds_k(t) = \left[\mathbf{M}_k^\top \mathbf{r}_k(t)c_k(t) + u_k(t) - \hat{d}_k(t) \right] dt + g_k(t)dW_k(t). \quad (22)$$

The change in the surplus aggregate level, $ds_k(t)$, is determined by the number of pieces that have successfully completed k stages of the manufacturing process, that are not defective, and are not consumed by stage $k + 1$. The first term on the right hand side of (22) represents the quantity produced which depends on the number of operational workstations for stage k . The term $u_k(t)dt$ is used to adjust the production rate. The term, $g_k(t)dW_k(t)$, is used to model the random fluctuations in the number of pieces produced. The effective demand or consumption term, $\hat{d}_k(t)dt$, is the consumption of the pieces produced by stage k by stage $k + 1$. The surplus aggregate level, $s_k(t)$, is dependent on the number of operational workstations, where the processes for the failure, repair, and preventive maintenance for the workstations is an *embedded Markov chain* (see Taylor and Karlin [15], for instance). These events describe the sojourn times for the discontinuous jumps in the surplus aggregate level. Hence, the surplus aggregate level is a piecewise continuous process whose discontinuous jumps are determined by the SDPP for the workstations.

The model presented here consists of two tiers. The top tier, is the upper level of management that is responsible for determining the production rates $d_k(t)$ for all k stages. The bottom tier, is the plant manager for the assembly floor who is responsible for maintaining the workstations, meeting the production goal set by the top tier, and consuming pieces that are in the buffers which is done by augmenting the production rates $d_k(t)$. The plant manager needs to consume pieces left in the buffers in a specified way, for example in a fixed amount of time or over the remaining production horizon. The effective demand, $\hat{d}_k(t)dt$, is used to meet the production goals of both tiers. Between stage i and $i + 1$ there is a buffer or storage area for pieces produced by stage i but not consumed by stage $i + 1$. Let $b_i(t)$

denote the number of these pieces which is determined by

$$b_i(t) = \left\{ \begin{array}{ll} \int_0^t [\mathbf{M}_i^\top \mathbf{r}_i(\tau) c_i(\tau) - \mathbf{M}_{i+1}^\top \mathbf{r}_{i+1}(\tau) c_{i+1}(\tau)] d\tau, & \text{if } i < k \\ \int_0^t [\mathbf{M}_i^\top \mathbf{r}_i(\tau) c_i(\tau) - d_i(\tau)] d\tau, & \text{if } i = k \end{array} \right\}, \quad (23)$$

note that $b_i(t) \geq 0$ for $i < k$ by design due to the construction of the production rate $c(t)$. The effective demand is used to eliminate pieces in the buffers in a specified way and is given by:

$$\hat{d}_i(t) = \left\{ \begin{array}{ll} d_i(t), & \text{if } i = 1 \\ d_i(t) + b_i(t)/\tau(t), & \text{if } 2 \leq i \leq k \end{array} \right\}, \quad (24)$$

where $\tau(t)$ is the amount of time used consume the pieces in the buffers that does not depend on the stage. Workstation failures can result in large buffer values, so upon repair a fixed amount of time is used to consume the pieces in the buffers. At the end of this time, a repair buffer event is generated. The effective demands are determined when there is an event which is a change in workstation status, repair buffer, and the start of the production run.

3.3. Cost Functional

The cost functional used is the *time-to-go* or *cost-to-go* form (6), using the state parameterized full quadratic instantaneous cost (7) that is motivated by a *zero inventory* or *Just in Time* manufacturing discipline (see Hall [9] and Bielecki and Kumar [3]) while utilizing minimum control effort. In this formulation, the cost functional employed is:

$$V[\mathbf{x}, \mathbf{s}, \mathbf{u}, t] = \frac{1}{2} (\mathbf{s}^\top S \mathbf{s}) (t_f) + \int_t^{t_f} \left[\frac{1}{2} \mathbf{u}^\top R_2(t; \mathbf{x}) \mathbf{u} + Q_1^\top(t; \mathbf{x}) \mathbf{s} \right] d\tau \quad (25)$$

with only the surplus aggregate level of the state used for the cost. The salvage cost, $S(t_f)$, is used to impose a penalty on surplus or shortfall of production at the end of the planning horizon. The diagonal matrix $Q_1(t; \mathbf{x})$ is the cost used to penalize shortfall and surplus production during the planning horizon, maintaining a strict regimen on when the consumable goods are to be produced and is parameterized by the state given by

$$Q_{1,kk}(t; \mathbf{x}) = \left\{ \begin{array}{ll} -\overline{Q_{1,k}^-}, & \text{if } s_k(t) < 0 \\ \overline{Q_{1,k}^+}, & \text{if } s_k(t) \geq 0 \end{array} \right\}, \quad (26)$$

where $\overline{Q_{1,k}^-}$ and $\overline{Q_{1,k}^+}$ are positive constant coefficients. The term $R_2(t; \mathbf{x})$ is used to enforce a minimum control effort penalty similar to that of (26).

3.4. Computational Considerations

To solve this optimal control problem, assume the regular or unconstrained control (12). The state space is partitioned into locally state independent subdomains and the solution for the nonlinear system of ordinary differential equations (13,14,15) is determined. This allows the plant manager of the manufacturing system to calculate the desired or ideal production rate and the physically realizable production rate. The production rate used is the physically realizable rate.

Let $n_k(t)$ denote the number of operational workstations on stage k and is given by $n_k(t) = \mathbf{1}^\top \mathbf{r}_k(t)$, where $\mathbf{1}$ is a $N_k \times 1$ vector whose elements are all 1. The regular controlled production level for stage k anticipates the effects of workstation failure, repair, and maintenance and small local effects is given by

$$c_k^{\text{reg}}(t) = \left\{ \begin{array}{ll} 0, & \text{if } n_k(t) = 0 \\ c_k(t) + u_k^{\text{reg}}(t)/(\mathbf{M}_k^\top \mathbf{r}_k(t)), & \text{if } n_k(t) > 0 \end{array} \right\}, \quad (27)$$

where $u_k^{\text{reg}}(t)$ is the regular control. Note that with the assumption of regular control, the surplus aggregate level will always be forced to be zero, therefore the regular controlled production level may not be physically realizable. The constrained controlled production level, $c_k^*(t)$, is the restriction of the regular controlled production level to be physically realizable and is given by

$$c_k^*(t) = \min[c_k^{\text{reg}}(t), c_k^{\text{max}}(t)], \quad (28)$$

where $c_k^{\text{max}}(t)$ is defined as

$$c_k^{\text{max}}(t) = \begin{cases} 1, & \text{if } k = 1 \\ \min [1, (c_{k-1}^*(t) \mathbf{M}_{k-1}^\top \mathbf{r}_{k-1}(t) + b_k(t)/\tau(t)) / (\mathbf{M}_k^\top \mathbf{r}_k(t))] , & \text{if } 1 < k \end{cases} \quad (29)$$

for $n_k(t) > 0$ where the maximum production rate, $c_k^{\text{max}}(t)$, is the minimum value of the physical production rate, 1.00 or full utilization, and production limitations that arise due to a shortfall of production from the previous stage due to either machine failure, maintenance, or defective pieces as well as clearing pieces in the buffers.

The computational algorithm for solving the optimal control problem for production scheduling uses the regular control. First the state space is partitioned into locally state independent subdomains, $\mathcal{D}_x = \bigcup_i \mathcal{D}_{x_i}$. It is assumed that the initial status for all workstations and the surplus aggregate level for all stages are known initially. The events of start of production, repair buffer, workstation repair, failure, and preventive maintenance are used to determine when the production rates for the manufacturing system need to be recalculated. The following steps are used to determine the production rates and the surplus aggregate levels for all stages for a given trajectory when an event occurs:

1. Based on the current state of the system the appropriate locally state independent subdomain is selected and the coefficients for the dynamics and cost functional are determined.
2. The effective demand rates are determined from (24).
3. The system of equations (13,14,15) is solved to determine $S(t)$, $\mathbf{D}(t)$ and $E(t)$ respectively.
4. The regular control, \mathbf{u}_{reg} is determined from (12).
5. The regular controlled production rates are determined by (27) for each stage.
6. The constrained controlled production rates, $c_k^*(t)$, are determined by (28) for each stage.
7. The surplus aggregate level for each stage k at the current time t is set to the cumulative difference between the production and the demand which is given by $s_k(t) = \int_0^t [\mathbf{M}_k^\top \mathbf{r}_k(\tau) c_k(\tau) - d_k(\tau)] d\tau$.
8. The constrained controlled production rates are used as the new rates for operational workstations on each stage, that is $c_k(t) = c_k^*(t)$.

4. Numerical Example for Production Scheduling

For numerical concreteness, consider a manufacturing system with $k = 2$ stages with a planning horizon of $T = 80$ hours. Let the initial surplus aggregate level for all stages be zero, the demand be 145 pieces per hour for all stages ($d_1(t) = d_2(t) = 145$), the total number of workstations, N_i , for each stage be 3 and 2, respectively, the Gaussian random fluctuations of production is assumed absent ($g_i(t) = 0$ for $i = 1$ and 2). The operational characteristics

for the workstations are summarized in the table below. During preventive maintenance and workstation failure no production occurs. Therefore, the moments for the moments for the state dependent Poisson processes in (18), (19), (17), and (21) are given by $\bar{Z}_{ki}^F = \bar{Z}_{ki}^R = \bar{Z}_{ki}^D = \bar{Z}_{ki}^M = 1$ with all covariances being 0. Assume that when the operational age of a workstation (21) is reset either due to a repair or preventive maintenance the operational age of the workstation is set to zero and that the aging process is based on the amount of time operational only. This implies that, $f(c_k(t), t)dt = 1$ and $H_{ki}^D(t) = H_{ki}^R(t) = \tau_{ki} - t - a_{ki}(\tau_{ki})$, where τ_{ki} is the time of the last reset (initial value is 0) and $a_{ki}(\tau_{ki})$ is viewed as a parameter that represents the age of the workstation at the last reset, which is zero for all $\tau_{ki} \neq 0$ and is specified in the table below for $\tau_{ki} = 0$. For repair buffer events let $\tau(t) = 4$, and $\tau(t) = T = t$ for all other events.

Stage k	Workstation i	Production Capacity, M_{ki} (pieces/hour)	Operational Age, $a_{ki}(0)$ (hours)	Mean Times (hours)			
				T_{ki}^F	T_{ki}^R	T_{ki}^M	T_{ki}^D
1	1	65	13	170	6	85	2
1	2	70	60	180	8	90	2
1	3	75	0	220	6	110	2
2	1	135	6	190	8	95	2
2	2	115	50	170	7	85	2

This manufacturing system consists of 20 local and 2 global state variables for a state of dimension 22. Define the local state vectors as $* (t) = [*_{11}(t), *_{12}(t), *_{13}(t), *_{21}(t), *_{22}(t)]^\top$, for the available production capacity $* = r$, operational status $* = o$, preventive maintenance status $* = m$, and current operational age $* = a$. Define the global state vector for the surplus aggregate level as $\mathbf{s}(t) = [s_1(t), s_2(t)]^\top$. The total state and control vectors are given by

$$\mathbf{X}(t) = [\mathbf{x}(t), \mathbf{s}(t)]^\top = [\mathbf{o}(t), \mathbf{m}(t), \mathbf{r}(t), \mathbf{a}(t), \mathbf{s}(t)]^\top, \quad \mathbf{u}(t) = [u_1(t), u_2(t)]^\top. \quad (30)$$

The cost functional used is (25) where the coefficient matrices are given by

$$S(t_f) = S_f = \begin{bmatrix} 0.14 & 0 \\ 0 & 0.17 \end{bmatrix}, \quad R(t) = \begin{bmatrix} 32000 & 0 \\ 0 & 32000 \end{bmatrix}, \quad \overline{Q_{1,1}^-} = 0.12, \quad \overline{Q_{1,2}^-} = 0.14, \\ \overline{Q_{1,1}^+} = 0.08, \quad \overline{Q_{1,2}^+} = 0.10.$$

By comparing the coefficients of (1) with the state equations for the manufacturing system (18), (19), (17), (21), and (22) the deterministic coefficients are given by

$$A(t) = \begin{bmatrix} 0_{20 \times 10} & 0_{20 \times 5} & 0_{20 \times 7} \\ 0_{2 \times 10} & M_A(t) & 0_{2 \times 7} \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0_{20 \times 2} \\ I_{2 \times 2} \end{bmatrix}, \quad C(t) = \begin{bmatrix} \mathbf{0}_{15 \times 1} \\ \mathbf{1}_{5 \times 1} \\ \hat{d}_1(t) \\ \hat{d}_2(t) \end{bmatrix}, \quad (31)$$

where

$$M_A(t) = \begin{bmatrix} M_{11}c_1(t) & M_{12}c_1(t) & M_{13}c_1(t) & 0 & 0 \\ 0 & 0 & 0 & M_{21}c_2(t) & M_{22}c_2(t) \end{bmatrix}.$$

Define the set $\gamma = \{11, 12, 13, 21, 22\}$ which is an index set for the stage and workstation, respectively. Define the diagonal matrix $\Delta_{*\gamma_i, c}$ such that the (i, i) component is given by $\delta_{*\gamma_i, c}$. The only *nonzero* stochastic process and

corresponding coefficient matrix given by

$$d\mathbf{P}_3(\mathbf{X}(t), t) = \begin{bmatrix} d\mathbf{P}^R(\mathbf{X}(t), t) \\ d\mathbf{P}^F(\mathbf{X}(t), t) \\ d\mathbf{P}^D(\mathbf{X}(t), t) \\ d\mathbf{P}^M(\mathbf{X}(t), t) \end{bmatrix}, \quad H_3(t) = \begin{bmatrix} \Delta_{o_{\gamma_i},0} & -\Delta_{r_{\gamma_i},1} & 0_{5 \times 5} & 0_{5 \times 5} \\ 0_{5 \times 5} & 0_{5 \times 5} & I_{5 \times 5} - \Delta_{m_{\gamma_i},1} & -\Delta_{r_{\gamma_i},1} \\ \Delta_{o_{\gamma_i},0} & -\Delta_{r_{\gamma_i},1} & I_{5 \times 5} - \Delta_{m_{\gamma_i},1} & -\Delta_{r_{\gamma_i},1} \\ -H_3^R(t) & 0_{5 \times 5} & -H_3^D(t) & 0_{5 \times 5} \\ 0_{2 \times 5} & 0_{2 \times 5} & 0_{2 \times 5} & 0_{2 \times 5} \end{bmatrix},$$

with $-H_3^R(t) = -H_3^D(t) = \text{diag}[\tau_{\gamma_i} - t - a_{\gamma_i}(\tau_{\gamma_i})]_{5 \times 5}$, where $\text{diag}[\mathbf{v}] = [v_i \delta_{i,j}]_{k \times k}$ is the diagonal matrix representation of the $k \times 1$ vector \mathbf{v} and the state dependent Poisson processes for $* \in \{R, F, D, M\}$ are given by $d\mathbf{P}^*(\mathbf{X}(t), t) = [d\mathbf{P}_{\gamma_i}^*(\mathbf{X}(t), t)]_{5 \times 1}$.

Using the above numerical values the algorithm presented in Section 3.4. can be used to determine the production rates for the manufacturing system. Consider the sample path trajectory described in the table below.

Event Time (hours)	Stage	Workstation	Event	Duration (hours)
15	2	2	failure	7
22	2		repair event	4
30	1	2	maintenance	2.5
72	1	1	maintenance	2

The constrained and regular controlled production rates for the manufacturing system are given in Figure 1. These production rates show the anticipation of workstation repair, failure, and maintenance. In Figure 2, the percent relative

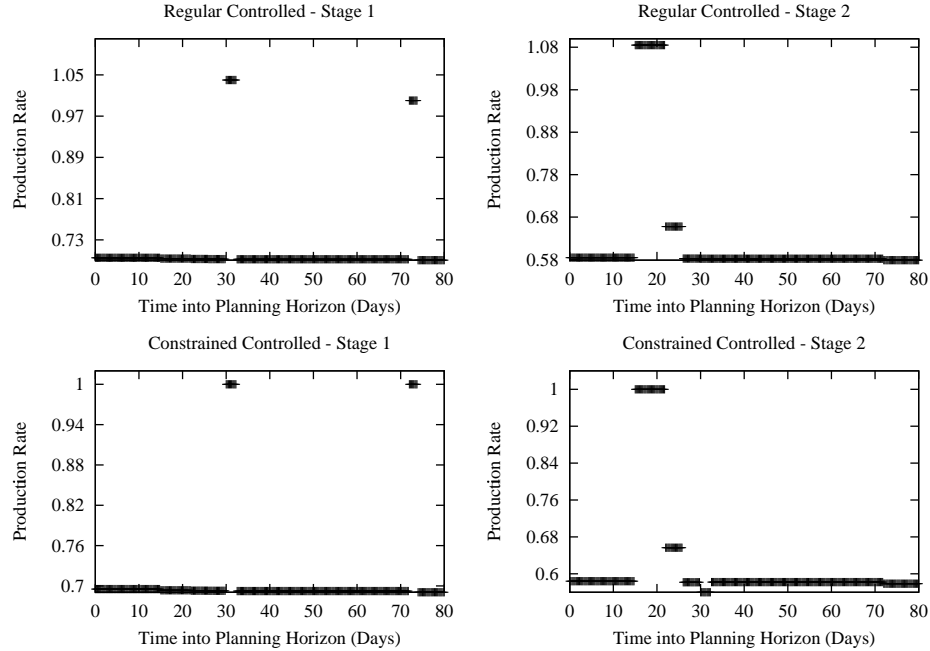


Figure 1: Regular and constrained controlled production rates for stages 1 and 2.

error is given. At the final time of the planning horizon the percent relative error is $(0.2252, 0.2252)^\top$. The results presented here required approximately 30 wall clock seconds to complete on a Sun Ultra 5, with a memory demand of

2.064 megabytes.

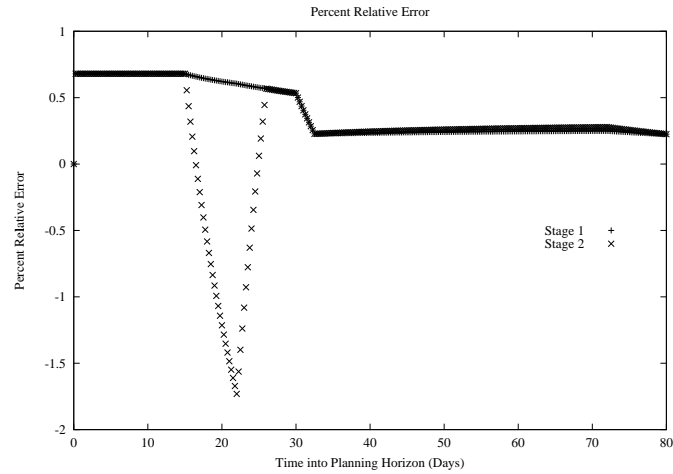


Figure 2: Percent relative error for stages 1 and 2.

5. Conclusions

The LQGP problem with state dependent Poisson processes forms a canonical theoretical and computational model that does not suffer from the curse of dimensionality. The inclusion of state parameterized coefficients allow for greater flexibility in the modeling of physical systems. The extension to a fully quadratic, state parameterized cost functional allows for the control to be selected in a very general way. The production scheduling of the multistage manufacturing system (MMS) is a real physical problem that exploits the functionality of the LQGP problem to solve a complex control problem with a large state space which requires a small amount of computational time. The MMS example illustrates the power and functionality of the LQGP problem presented in this paper.

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