Optimal Consumption and Portfolio Control for Jump–Diffusion Stock Process with Log–Normal Jumps (Corrected)*

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Abstract

A computational solution is found for a optimal consumption and portfolio policy problem in which the underlying stock satisfies a geometric jump–diffusion in which both the diffusion and jump amplitude are log–normally distributed. The optimal objective is to maximize the expected, discounted utility of terminal wealth and the cumulative discounted utility of instantaneous consumption. The jump–diffusion allows for a more realistic distribution, skewed toward negative jumps and having leptokurtic behavior in which the tails are thicker so that the distribution is more slender around the peak than normal. Computational issues pertinent to jump–diffusion calculations are discussed. This is a **corrected** version of the published paper.

1. Introduction

The Black–Scholes [2] option pricing model based upon geometric Brownian motion has been widely used in spite of its deficiencies. However, many financial engineers have tried to correct the deficiences of this basic model. One big deficiency is that Brownian motion, having continuous sample paths, lacks the sudden jumps in value of real financial instruments. Merton [8, Chapter 9] applied discontinuous sample path Poisson processes, along with Brownian motion processes, i.e., jump–diffusions, to the problem of pricing options. Merton derived several extensions of the already classical diffusion theory of Black–Scholes minimizing the portfolio variance for jump–diffusion models using techniques modified from those used to derive the Black–Scholes formulae.

Prior to the Black–Scholes model, Merton [8, Chapter 5-6] analyzed the optimal consumption and investment portfolio with either geometric Brownian motion or Poisson noise and illustrated explicit solutions for constant risk–aversion utility. In [8, Chapter 4] , Merton also examined constant risk–aversion problems. Sethi and Taksar [11] present corrections to certain formulae Merton's finite horizon consumption–investment model. Merton [8, Chapter 6] revisited the problem in his continuous–time and reprint finance book, correcting his earlier work by adding an absorbing boundary condition at zero wealth and using other techniques. Wilmott's [13] presents a good discussion on hedging with jump–diffusion models in finance, coming to the conclusion that for a single option *perfect risk–free hedging is impossible when there are jumps in the underlying.*

In the paper of Hanson and Westman [6], a complex optimal portfolio and consumption policies problem was solved computationally. The financial model was modified from a theoretical important event model proposed by Rishel [10] that is an optimal portfolio and consumption model for a portfolio of stocks and a bond. The stock prices are dependent on both scheduled (deterministic) and unscheduled (stochastic) jump external events in an environment of geometric jump–diffusion processes. The jumps affect both the stock prices directly or indirectly through parameters. The scheduled jumps are actually quasi–deterministic, in that the timing of the event is deterministic, but the magnitude of the jump is random. The computations were illustrated for a simple discrete jump model, such that both stochastic and quasi–deterministic jump magnitudes were heuristically estimated discretely distributed single negative or positive jumps. Motivation for this quasi–deterministic process are the more or less monthly announcements of the Federal Open Market Committee [3], but the response of the market to changes in Federal Funds Rate or Federal Discount Rate is not too predictable. This quasi–deterministic process might be called the *Greenspan Process*. The current paper applies log–normal jump amplitude distribution to the optimal portfolio and consumption problem, without quasi–deterministic processes.

It is well known that the distribution daily investment returns, i.e., the relative change in the instrument, can have

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fatter tails and be more slender near the mode than the corresponding normal distribution with the same mean and variance as the investment sample. Such a property is called *leptokurtic* since the kurtosis of the distribution is larger than that of the normal distribution. Also, the negative market tails tend to be fatter than the positive tails, so that the skew of the distribution will be negative.

In his discontinuous returns paper, Merton [8, Chapter 9] considers the case of option pricing modeled by a jump–diffusion process where the jumps have a log–normal distribution in one important example. Andersen and Andreasen [1] treat the log–normal jump–diffusion option pricing problem in much more detail, both analytically through forward partial integral–differential equations and numerically mainly through alternating direction implicit methods.

Hanson and Westman [4] derived the probability density of the log–normal jump–diffusion using probabilistic methods by treating the log–return process as a triad, X+Y*Z, random variables. The product, Y*Z, represents the space–time Poisson process, the product of the log–normal random jump amplitude and the Poisson jump counting process. Once the distribution of the product is determined, then sum of the log–normal diffusion and the space–time Poisson follows from the well–known convolution of distributions of the summands. The parameters of this log–normal jump–diffusion were estimated by fitting the distribution Standard and Poor's 500 (S&P500) log–returns for daily closings based upon minimum variance between the S&P500 empirical distribution and the theoretical discretized log–return density of the corresponding log–normal jump–diffusion.

In this paper, these jump–diffusion parameter values, supplemented by additional economic data are applied to the optimal portfolio and consumption control with constant relative risk–adverse utilities problem. In Section 2, the optimal portfolio and consumption control problem is defined and reduced to a stochastic dynamic programming problem. In Section 3, the problem is then simplified by transforming to the canonical constant relative risk–adverse utility problem with corresponding elimination of the dimensional computational complexity of the wealth state space. In Section 4, computational results for the optimal value, portfolio and consumption are presented. Finally, the concluding remarks are in Section 5.

2. Optimal Portfolio and Consumption Control Problem

Let S(t) be the price of a single stock or mutual fund that satisfies the Markov, geometric, log–normal jump–diffusion stochastic differential equation (SDE),

$$dS(t) \quad = \quad S(t)\left[\mu_d dt + \sigma_d dZ(t) + JdP(t)\right] \,, \qquad (1)$$

$S(0) = S_0$, $S(t) > 0$, where μ_d is the mean appreciation return rate, σ_d is the diffusive volatility, $Z(t)$ is a continuous, one–dimensional Brownian motion process, J is a random jump amplitude with log–return mean μ_j and variance σ_j^2, and $P(t)$ is a discontinuous, one–dimensional, standard Poisson process with jump rate λ. Here, we will assume that the jump–diffusion parameters μ_d, σ_d, μ_j, σ_j and λ are constant. The stochastic processes $Z(t)$ and $P(t)$ are Markov and pairwise independent. The jump amplitude process J, given a Poisson jump in time, is also independently distributed.

The continuous, diffusion process $Z(t)$ is standard, so has infinitesimal moments $\mathrm{E}[dZ(t)] = 0$ and $\mathrm{Var}[dZ(t)] = dt$. **The discontinuous space–time jump process $JdP(t)$ is just a symbol and can be defined by a stochastic integral of Poisson random measure $\mathcal{P}(dt, dq)$ or as a sum of $dP(t)$ jumps of a compound Poisson process**,

$$\mathbf{JdP(t)} = \int_{\mathcal{Q}} \mathbf{J(q)}\mathcal{P}(\mathbf{dt}, \mathbf{dq}) = \sum_{\mathbf{i=1}}^{\mathbf{dP(t)}} \mathbf{J(Q_i)}$$

with $\mathrm{E}[\mathcal{P}(dt, dq)] = \lambda dt \phi_Q(q) dq$ **and** $\sum_{\mathbf{i=1}}^{\mathbf{0}} \mathbf{J(Q_i)} \equiv \mathbf{0}$**, where** q **is the mark for the jump amplitude process corresponding to the underlying random variable** $\mathbf{Q_i}$, such that $-1 < J(q) < \infty$ so that one jump does not make the underlying worthless, and where $\phi_Q(q)$ is the Poisson mark density, providing it exists on the mark space $\mathcal{Q}$. The infinitesimal moments of the jump process are $\mathrm{E}[JdP(t)] = \mathrm{E}[J]\lambda dt$ and $\mathrm{Var}[\mathbf{JdP(t)}] = \mathrm{E}[\mathbf{J^2}]\lambda \mathbf{dt}$.

Before describing the mark distribution in more detail, it is helpful to first transform the stock price SDE (1) to the SDE of the stock log–returns using the stochastic chain rule for Markov processes in continuous time,

$$d[\ln(S(t))] \quad = \quad \mu_{ld} dt + \sigma_d dZ(t) + \ln(1 + J(Q))dP(t) \,, \qquad (2)$$

where the log–diffusion drift $\mu_{ld} \equiv \mu_d - \sigma_d^2/2$ includes an Itô calculus shift of the mean appreciation rate by the diffusion coefficient and the log–return jump amplitude is the logarithm of the relative post–jump amplitude $\ln(1 + J(Q))$. This log–return SDE (2) is the model that was compared in [4] to the S&P500 log–returns, since they are preferred financial investment metric as a measure of the relative changes in investment value, as opposed to the absolute change of the stock price represented by the geometric jump–diffusion SDE in (1).

Since $J > -1$, let the mark process be the log–return jump amplitude $Q = \ln(1 + J)$, which has the inverse $J(Q) = \exp(Q) - 1$, on the mark space $\mathcal{Q} = (-\infty, +\infty)$. Further, on the full infinite domain, the natural choice for the mark density is the normal density, $\phi_n(q; \mu_j, \sigma_j^2)$, so

$$\phi_Q(q) = \phi_n(q; \mu_j, \sigma_j^2) \equiv \frac{\exp(-(q - \mu_j)^2/(2\sigma_j^2))}{\sqrt{2\pi\sigma_j^2}} \,, \qquad (3)$$

having a mean $E[Q] = \mu_j$ and variance $\mathrm{Var}[Q] = \sigma_j^2$. Hence, $J(Q) + 1$ is log–normally distributed. The moments of the log–return differential are

$$M_1^{(jd)} \quad \equiv \quad \mathrm{E}[d[\ln(S(t))]] = (\mu_{ld} + \lambda\mu_j)dt \,, \qquad (4)$$

$$M_2^{(jd)} \equiv \mathrm{Var}[d[\ln(S(t))]] = (\sigma_\mathbf{d}^2 + \lambda(\sigma_\mathbf{j}^2 + \mu_\mathbf{j}^2))\mathbf{dt}\,. \quad (5)$$

The log–return is of primary interest, as the investor is interested in the per cent or relative change in a portfolio and the log–return is the continuous limit of the relative change.

Let the portfolio consist of one riskless asset, say a bond, with price at time t of $B(t)$ dollars and one risky asset, a stock or mutual fund, with price $S(t)$ at time t. Further, let $U_0(t)$ be the fraction of the instantaneous portfolio change in the bond investment at time t and $U_1(t)$ be the fraction of the change in the stock investment, such that the sum adds up to the whole portfolio change in investment: $U_0(t)+U_1(t) = 1$. The instantaneous bond fraction $U_0(t) \geq 0$, i.e., must be non–negative. It will also be assumed that the stock fraction $U_1(t)$ will be non–negative to keep the multitude of special cases to a minimum, so that $0 \leq U_1 \leq 1$. The bond is assumed to satisfy a deterministic exponential process

$$dB(t) = rB(t)dt\,, \quad B(0) = B_0\,. \quad (6)$$

with the bond price continuously compounded at a fixed rate of interest, r.

The wealth process at time t changes due to changes in the portfolio fraction depending on the relative change in portfolio prices less instantaneous consumption of wealth:

$$\begin{aligned} dW(t) &= W(t)\left[U_0(t)\frac{dB(t)}{B(t)} + U_1(t)\frac{dS(t)}{S(t)}\right] - C(t)dt \\ &= W(t)\left[rdt + U_1(t)\left\{(\mu_d - r)dt \right.\right. \\ &\quad \left.\left. +\ \sigma_d dZ(t) + JdP(t)\right\}\right] - C(t)dt\,, \end{aligned} \quad (7)$$

where $C(t)$ is the instantaneous rate of consumption, assumed to be non–negative as well as constrained relative to wealth, i.e., $0 \leq C(t) \leq C^{(0)}_{\max}W(t)$ and $U_0(t) = 1 - U_1(t)$ has been eliminated by the fraction constraint. At the ℓth Poisson jump time T_ℓ, the wealth jumps by $[W](T_k) \equiv W(T_k^+) - W(T_k^-) = U_1(T_k^-)J(q_k)W(T_k^-)$, for $k = 1, 2, 3, \ldots$ jumps with random amplitude $J(q_k) = e^{q_k} - 1$, where q_k is a normally distributed mark variable at the kth jump with mean μ_j and variance σ_j^2.

The investor's objective is to maximize the conditional, expected current value of the discounted utility $\mathcal{U}_f(w)$ of terminal wealth at the end of the investment terminal time T and the discounted utility of instantaneous consumption, i.e.,

$$\begin{aligned} v^*(t, w) &= \max_{\{u,c\}[t,T)} \left[E\left[e^{-\beta(T-t)}\mathcal{U}_f(W(T)) \right.\right. \\ &\quad \left.\left. + \int_t^T e^{-\beta(\tau-t)}\mathcal{U}(C(\tau))d\tau\right]\right]\,, \end{aligned} \quad (8)$$

conditioned on the state–control set $\mathcal{C} = \{W(t) = w, U_1(t) = u, C(t) = c\}$, where $0 \leq t < T$, $0 \leq u \leq 1$, $0 \leq c \leq C^{(0)}_{\max}w$ for non–negative consumption feasibility with maximal relative limits, $w \geq 0$ for non–negative wealth feasibility, and β is the fixed discount rate. Thus, the instantaneous consumption $c = C(t)$ and stock portfolio fraction $u = U_1(t)$ serve as control variables, while the wealth $w = W(t)$ is the state variable. Bellman's Principle of Optimality has the form,

$$\begin{aligned} v^*(t, w) &= \max_{\{u,c\}[t,t+dt)} \left[E_{[t,t+dt)}\left[\mathcal{U}(c)dt \right.\right. \\ &\quad \left.\left. +\ (1 - \beta dt)v^*(t + dt, w + dW(t))\right]\right]\,, \end{aligned} \quad (9)$$

conditioned on set $\mathcal{C}$, for sufficiently small dt when $0 \leq t \leq T$, subject to the zero wealth absorbing boundary condition to avoid the possibility of arbitrage [8],

$$v^*(t, 0^+) = \mathcal{U}_f(0)e^{-\beta(T-t)} + \mathcal{U}(0)(1 - e^{-\beta(T-t)})/\beta \quad (10)$$

and assuming that the consumption must be zero when the wealth is zero. The bequest or terminal wealth condition $v^*(T, w) = \mathcal{U}_f(w)$. must also be satisfied and as a final condition means that the problem will be a final value problem rather than an initial value problem.

Assuming the $v^*(t, w)$ is continuously differentiable in t and twice continuously differentiable in w, then the stochastic dynamic programming equation (see Kushner [7] for early Poisson jump versions) follows from an application of the Itô stochastic chain rule to the principle of optimality form (9):

$$\begin{aligned} 0 &= v_t^*(t, w) - \beta v^*(t, w) + \mathcal{U}(c^*) \\ &\quad + \left[(r + (\mu_d - r)u^*)w - c^*\right] v_w^*(t, w) \\ &\quad + \frac{1}{2}\sigma_d^2(u^*)^2 w^2 v_{ww}^*(t, w) \\ &\quad + \lambda \int_{-\infty}^{+\infty} \left[v^*(t, (1 + J(q)u^*)w) \right. \\ &\quad \left. -\ v^*(t, w)\right] \phi_n(q; \mu_j, \sigma_j^2)dq\,, \end{aligned} \quad (11)$$

where $u^* = u^*(t, w) \in [0, 1]$ and $c^* = c^*(t, w) \in [0, C^{(0)}_{\max}w]$ are the optimal controls if they exist, while $v_w^*(t, w)$ and $v_{ww}^*(t, w)$ are the partial derivatives with respect to wealth w when $0 \leq t < T$. Non–negativity of wealth implies an additional consistency condition for the control since the jump in wealth argument $(1 + J(q)u^*)w$ requires that $1 + J(q)u \geq 0$,which is requires $0 \leq u \leq 1$.

When the maximum in (8) is unconstrained and is attainable, then the controls are the regular controls $u_{\text{reg}}(t, w)$ and $c_{\text{reg}}(t, w)$, which are given implicitly, provided sufficient differentiability in c and u, by the dual critical conditions, $\mathcal{U}'(c_{\text{reg}}(t, w)) = v_w^*(t, w)$ for consumption and

$$\begin{aligned} &\sigma_d^2 w^2 v_{ww}^*(t, w)u_{\text{reg}}(t, w) = -(\mu_d - r)wv_w^*(t, w) \\ &\quad -\lambda w \textstyle\int_{-\infty}^{+\infty} J(q)v_w^*(t, (1 + J(q)u_{\text{reg}}(t, w))w) \\ &\quad \cdot\phi_n(q; \mu_j, \sigma_j^2)dq \end{aligned}$$

for portfolio policies.

3. Constant Relative Risk–Adverse Utility Canonical Problem Reduction

Assuming the investor is risk adverse, the utilities will be taken to be Constant Relative Risk–Aversion (CRRA) power utilities with the same power for wealth and consumption:

$$\mathcal{U}(x) = \mathcal{U}_f(x) = x^\gamma/\gamma\ , \quad x \geq 0, \quad 0 < \gamma < 1\ , \tag{12}$$

which is consistent with the non–negativity requirements of w and c. The CRRA utilities are a special case of Hyperbolic Absolute Risk Aversion (HARA) utility treated by Merton [8, Chapter 4-6].

These power utilities lead for the optimal consumption and portfolio problem to a canonical reduction of the stochastic dynamic programming PDE problem to a simpler ODE problem. The optimal value function has a solution that separates the wealth state variable from the time dependence,

$$v^*(t, w) = \mathcal{U}_f(w) v_0(t)\ , \tag{13}$$

where the wealth dependence is given explicitly and the time function is to be determined. Since $\mathcal{U}_f(0^+) = \mathcal{U}(0^+) = 0$ from (12), the absorbing boundary (10), i.e., $v^*(t, 0^+)$, is automatically satisfied.

Further, the regular consumption control is a linear function of the wealth,

$$c_{\text{reg}}(t, w) \equiv w \cdot c_{\text{reg}}^{(0)}(t) = w/v_0^{1/(1-\gamma)}(t)\ , \tag{14}$$

since $v_w^*(t, w) = \mathcal{U}'(w)v_0(t) = \mathcal{U}'(c_{\text{reg}}(t, w))$ and $\mathcal{U}'(x) = x^{\gamma-1}$ using (12). The regular stock fraction reduces to a wealth and time independent control, $u_{\text{reg}}(t, w) = u_{\text{reg}}^{(0)}$, defined implicitly by

$$\begin{aligned} u_{\text{reg}}^{(0)} &= G(u_{\text{reg}}^{(0)}) \equiv \frac{1}{(1-\gamma)\sigma_d^2}\left[\mu_d - r + \lambda I_1(u_{\text{reg}}^{(0)})\right], \quad (15) \\ I_1(u) &\equiv \int_{-\infty}^{+\infty} J(q)\left(1 + J(q)u\right)^{\gamma-1} \phi_n(q; \mu_j, \sigma_j^2) dq, \end{aligned}$$

since $w^2 v^*(t, w) = \gamma(\gamma-1)\mathcal{U}(w)v_0(t)$ and $\mathcal{U}((1+Ju)w) = \mathcal{U}(1 + Ju)\mathcal{U}(w)$. This wealth independent property of the regular stock fraction is essential for the separability of the optimal value function (13). Since (15) only defines $u_{\text{reg}}^{(0)}$ implicitly in fixed point form, $u_{\text{reg}}^{(0)}$ must be found by iteration and a good choice is Newton's method, a fast and accurate fixed point method, assuming $|G'(u_k) - 1| > \varepsilon > 0$ and

$$\begin{aligned} u_{k+1} &= u_k - (G(u_k) - u_k)\,/\,(G'(u_k) - 1)\ , \\ G'(u) &= \tfrac{\lambda}{(1-\gamma)\sigma_d^2} I_1'(u) - 1\ , \\ I_1'(u) &= (1-\gamma)\textstyle\int_{-\infty}^{+\infty} J(q)^2 \left(1 + J(q)u\right)^{\gamma-2} \\ &\quad \phi_n(q; \mu_j, \sigma_j^2) dq\ . \end{aligned}$$

The integrals are efficiently approximated by a 3–point Gauss–Statistics quadrature [12] (a general Gaussian quadrature that, with a log–normal jump density, is a variation of the Gauss–Hermite quadrature, except with simpler nodes $\{-\sqrt{3}, 0, \sqrt{3}\}$ and weights $\{1/6, 2/3, 1/6\}$, having fifth degree polynomial precision), using the standardized normal density upon transforming the jump density.

The optimal controls, when there are constraints, are given in piecewise form as $c^*(t, w)/w = c_0^*(t) = \max[\min[c_{\text{reg}}^{(0)}(t), C_{\max}^{(0)}], 0]$, provided $w > 0$, and $u^* = \max[\min[u_{\text{reg}}^{(0)}, 1], 0]$, independent of w and t along with $u_{\text{reg}}^{(0)}$.

Substitution of the separable power solution (13) and the regular controls in (14-15) into the stochastic dynamic programming equation (11), leads to an ODE,

$$0 = v_0'(t) + (1-\gamma)\left(g_1(u^*)v_0(t) + g_2(t)v_0^{\frac{\gamma}{\gamma-1}}(t)\right)\ , \tag{16}$$

$$\begin{aligned} g_1(u) &\equiv \frac{1}{1-\gamma}\left[-\beta + \gamma\left(r + u(\mu_d - r)\right)\right. \\ &\quad \left. - \frac{\gamma(1-\gamma)}{2}\sigma_d^2 u^2 + \lambda(I_2(u) - 1)\right] \\ g_2(t) &\equiv \frac{1}{1-\gamma}\left[\left(\frac{c_0^*(t)}{c_{\text{reg}}^{(0)}(t)}\right)^\gamma - \gamma\left(\frac{c_0^*(t)}{c_{\text{reg}}^{(0)}(t)}\right)\right], \quad (17) \\ I_2(u) &\equiv \gamma\int_{-\infty}^{+\infty} \mathcal{U}(1 + J(q)u)\phi_n(q; \mu_j, \sigma_j^2) dq\ , \end{aligned}$$

for $0 \leq t < T$. The coupling of $v_0(t)$ to the time dependent part of the consumption term $c_{\text{reg}}^{(0)}(t)$ in $g_2(t)$ (17), and the relationship of $c_{\text{reg}}^{(0)}(t)$ to $v_0(t)$ in (14), means that the ODE (16) is actually highly nonlinear and thus (16) is only of Bernoulli type implicitly.

The implicit Bernoulli equation (16) can be transformed to a linear differential equation by using $\theta(t) = v_0^{1/(1-\gamma)}(t)$, to obtain, $0 = \theta'(t) + g_1(u^*)\theta(t) + g_2(t)$, whose general solution can be inverse transformed to the particular solution for the separated time function implicitly given by

$$\begin{aligned} v_0(t) &= \theta^{1-\gamma}(t) = \left[e^{-g_1(u^*)(T-t)}\left(1\right.\right. \quad (18) \\ &\quad \left.\left. + \int_t^T g_2(\tau) e^{g_1(u^*)(T-\tau)} d\tau\right)\right]^{1-\gamma}, \end{aligned}$$

using the final condition $v_0(T) = 1$, since $v_0(t)$ depends on $c_{\text{reg}}^{(0)}(t)$ in (17), while $c_{\text{reg}}^{(0)}(t)$ depends on $v_0(t)$ in (14). Hence, both $v_0(t)$ and $c_{\text{reg}}^{(0)}(t)$ must be found by computational iteration, so a much more efficient and backward integration form upon transformation and decomposition, is used and is given by $v_0(t - \Delta t) = \theta^{1-\gamma}(t - \Delta)$ where $\theta(t - \Delta t) = \omega_1^*(-\Delta t)(\theta(t) + \int_{t-\Delta t}^t g_2(\tau)\omega_1^*(t - \tau)d\tau)$ and $\omega_1^*(t) \equiv \exp(g_1(u^*)t)$. An exponentially weighted, 2–point trapezoidal rule with weight $\omega_1^*(t - \tau)$ is used to approximate the short integral over $g_2(\tau)$ (17), evaluated at the current value t and at the backward iterate $t - \Delta t$, iterating until the last change, $|v_0(t - \Delta t) - v_0(t)|$, is smaller in than a prescribed tolerance.

The solution for the optimal value function is $v^*(t, w) = \mathcal{U}_f(w)v_0(t)$, requiring only multiplication by the utility. The optimal portfolio fraction control is the constant u^* under control constraints, using solutions from (15). The optimal consumption control $c^*(t,w)$ is time and wealth dependent and under constraints from (14). However, the feasibility of calculating the solution by iteration is extremely good.

4. Computational Finance Results

In our paper [4] the log–normal density of the log–returns is derived and proven by basic probabilistic methods, as summarized and **corrected here** in the following result:

Theorem: The probability density for the log–normal jump–diffusion log–return differential $d[\ln(S(t))]$ specified in the SDE (2) is

$$\phi_{d\ln(S(t))}(z) = \sum_{k=0}^{\infty} p_k(\lambda dt) \cdot\phi_{\mathbf{n}}(\mathbf{z};\mu_{\mathbf{ld}}\mathbf{dt}+\mu_{\mathbf{j}}\mathbf{k},\sigma_{\mathbf{d}}^{\mathbf{2}}\mathbf{dt}+\sigma_{\mathbf{j}}^{\mathbf{2}}\mathbf{k}), \quad (19)$$

$-\infty < z < +\infty$, where $p_k(\lambda dt) = \exp(-\lambda dt)(\lambda dt)^k/k!$ for $k = 0, 1, 2, \ldots$, is the well–known Poisson distribution with parameter λdt, and the normal density ϕ_n is defined above in (3).

The proof relies on the fact that the density for a sum of random variables is a nested series of convolutions of the component densities. In the special case of a jump-diffusion where the Poisson jump amplitudes are normally distributed as is the diffusion process, for each jump count k the convoluted density is also normally distributed such that the mean is the sum of the means and the variance is the sum of the variances. For total density of the jump-diffusion is the sum of all the Poisson jump counts weighted by the Poisson distribution according to the law of total probability.

The log–normal jump–diffusion density (19) was fit in [4] to realistic data. The 1657 daily closings of the Standard and Poor's 500 (S&P500) stock index from 1995 to July 2001 were used from data available on–line [14]. The S&P500 data is an example of one large mutual fund rather than a single stock but has the advantage of not being biased severely to the extremes of any one stock. The data was transformed into changes in the natural logarithm of the index closings, $\Delta[\ln(SP_i)] \equiv \ln(SP_{i+1}) - \ln(SP_i)$ for $i = 1, \ldots, 1656$ points. Using 50 bin histograms the empirical S&P500 log–return data was compared to the corresponding histogram for the theoretical log–return jump–diffusion density $\phi_{d\ln(S(t))}$ in (19). The five parameter set of unknowns, $\{\mu_d, \sigma_d^2, \mu_j, \sigma_j^2, \lambda dt\}$ was to be reduced to a more manageable set of three to avoid large fitting errors by constraining both distributions to have the same mean (M_1) and variance (M_2), selecting the diffusive parameters for elimination:

$$\begin{aligned} \sigma_d^2 &= (\mathbf{M_2} - \lambda\mathbf{dt}(\sigma_{\mathbf{j}}^{\mathbf{2}} + \mu_{\mathbf{j}}^{\mathbf{2}}))/\mathbf{dt}\,, \\ \mu_d &= (M_1 - \lambda dt\mu_j)/dt\,. \end{aligned} \quad (20)$$

The reduced set $\{\mu_j, \sigma_j^2, \lambda dt\}$ was then found by minimizing the variance between the bins of the two histograms. Due to the complexity of the jump–diffusion density and the need to keep finance methods simple, a multi–dimensional modification of *Golden Section Search* was derived that needs no derivatives and searches beyond the current range when a local minimum is not found in the current search hypercube [5]. In addition, hypercube constraints were implemented so that the free model parameters $\{-\mu_j, \sigma_j^2, \lambda dt\}$ would remain non–negative and be bounded. The final parameter results reported in [4] are

$$\begin{aligned} \mu_d &\simeq 0.2712\,, \quad \sigma_d^2 \simeq 0.01048\,, \\ \mu_j &\simeq -0.0007474\,, \quad \sigma_j^2 \simeq 0.00007812\,, \quad \lambda \simeq 161.7\,. \end{aligned} \quad (21)$$

The return time, $dt \simeq 0.003964$, is the reciprocal of the average number of trading days per year or 252.3 days. A comparison of the coefficients of skew and kurtosis are given in Table 1.

Table 1: Comparison of coefficients of skew and kurtosis.

Distribution	Coeff. Skew $\eta_3 = M_3/M_2^{1.5}$	Coeff. Kurtosis $\eta_4 = M_4/M_2^2$
S&P500	-0.2867	6.862
Jump-Diffusion	-0.2114	8.082
Normal	+0.0000	3.000

Additional economic parameters are the average rate $r = 7.054\%$ for Moody AAA bonds, the average discount rate $\beta = 4.617\%$ from the Federal Reserve Bank, $\gamma = 0.20$ is taken as the common power of the CRRA terminal wealth and consumption utilities, and $C_{\max}^{(0)} = 0.75$ as an upper bound on the consumption relative to wealth. Some computational parameters are $W_{\max} = 100$ as a finite bound for wealth output representation only divided into $N_w = 20$ sub–intervals and $T = 1$ trading year is the investment terminal time divided into 100 subdivisions. The regular stock fraction control, found once and for all as a constant, independent of time and wealth, is $u_{\text{reg}}^{(0)} \simeq 4.62$, so that the optimal control $u^* = 1.00$ constrained on $[0.0, 1.00]$, taking only 3 iterates to get $u_{\text{reg}}^{(0)}$ to 3 significant digits.

For financial engineering applications, it is very important to have very feasible finance computations for quick and accurate approximate solutions. Hence, these computations were mainly carried out in the popular code development system of MATLAB™ [9] due to its facility for developing rapid prototype solutions.

In Figure 1, the optimal value $v^*(t,w)$ solution is shown in three–dimensions versus the wealth w in dollars and t in trading years. For fixed time, the optimal value follows the CRRA utility of wealth template. However, the scaling of the optimal value function and the dependence on time t for fixed wealth w depends on the separated time solution $v_0(t)$. Since the wealth utility $\mathcal{U}(w) = w^\gamma/\gamma$ is badly behaved due to the non–differentiability at $w = 0$ if $0 < \gamma < 1$, the wealth mesh has been transformed to constant intervals in w^γ, rather than on w itself. The way this stochastic dynamic programming solution is interpreted is that given a time t and wealth w position, than the optimal value of the portfolio is $v^*(t, w)$ using the optimal stock fraction policy of u^* given above. Since the problem is autonomous due to time–independent parameters, the results are also valid for investment terminal times between $t = 0.0$ and $T = 1.0$.

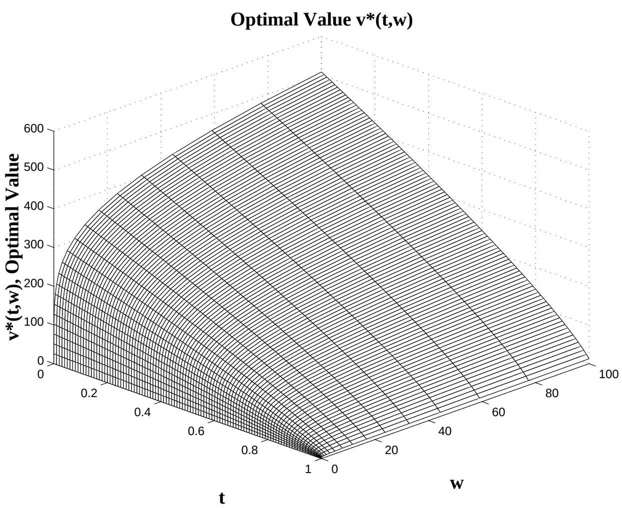

Figure 1: Optimal expected value approximation $v^*(t, w)$ versus time t and wealth w in numerical results for CRRA model.

In Figure 2, the computational approximation of the optimal consumption policy or control $c^*(t, w)$ is displayed versus the time t in trading years and the wealth w in dollars using the CRRA power utility model. Recall that $c^*(t, w)$ is linear in the wealth w, but inversely proportional to the separated optimal value time function $v_0(t)$ to the power $1/(1 - \gamma) = 1.25$ here when $\gamma = 0.2$. Hence, lines constant in time are straight lines, while the dependence in time t for fixed wealth w in $[0, 100]$ are roughly the reciprocal of $v_0(t)$, i.e., $v_0^{-1.25}(t)$.

5. Conclusions

The log–normal jump–diffusion distribution has been demonstrated on the canonical optimal portfolio and consumption control problem. Computational techniques are presented for handling the iterations for implicitly defined solutions such as the optimal stock fraction policy u^* and the coupled optimal value separated time function $v_0(t)$ and the optimal consumption policy c^*. Also, the Gauss–Statistics quadrature for handling the log–normal jump amplitude integral has been used, but this technique is also useful for other jump distribution by using the appropriate standardized distribution.

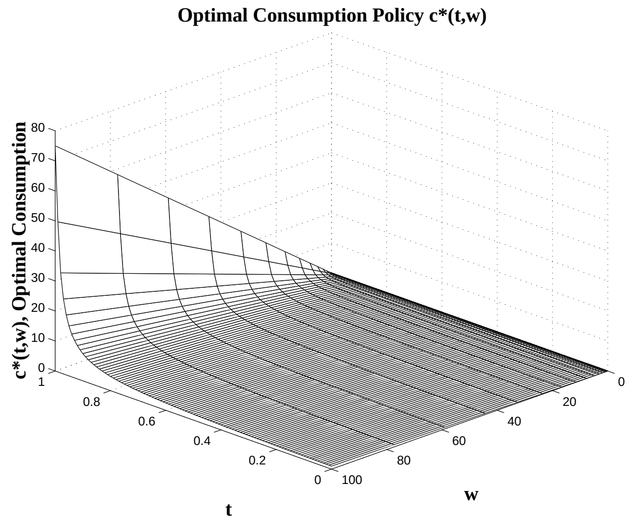

Figure 2: Optimal consumption policy approximation $c^*(t, w)$ versus time t and wealth w in numerical results for CRRA model.

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