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Jump–Diffusion Stock-Return Model with Weighted Fitting of Time-Dependent Parameters [1]

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Abstract

This paper treats jump-diffusion processes in continuous time, with emphasis on the jump-amplitude distributions, developing more appropriate models using parameter estimation for the market. The proposed method of parameter estimation is weighted least squares of the difference between theoretical and experimental bin frequencies, where the weights or reciprocal variances are chosen as by the theory of jump-diffusion simulation applied to bin frequencies. The empirical data is taken from a decade of Standard & Poor 500 index of stock closings and are viewed as one moderately large simulation. The new developments are the combined use of uniform jump-amplitude distributions, least squares weights and time-varying market parameters, introducing more realism into the model, a Log-Normal-Diffusion, Log-Uniform-Jump financial market model. The optimal parameter estimation is highly nonlinear, computationally intensive, and the optimization is with respect to the three parameters of the log-uniform jump distribution, while the diffusion parameters are constrained by the first two moments of the S&P500 data.

1. Introduction

The actual distribution of daily log-returns for financial market data differ in significant ways from the ideal log-normal diffusion process as assumed in the Black-Scholes model [1] and other option pricing models. The log-returns, the log-differences between two successive market closings, approximate the logarithm of the relative change. The most significant difference is that actual log-returns exhibit occasional large jumps in value, whereas the diffusion process in Black-Scholes [1] is continuous. A second difference is that the empirical log-returns are usually negatively skewed, since the crashes are likely to be larger or more numerous than the positive jumps, whereas the normal distribution associated with the diffusion process is symmetric. Thus, the coefficient of skew [2] is negative, $\eta_3 \equiv M_3/(M_2)^{1.5} < 0$, where M_2 and M_3 are the 2nd and 3rd central moments of the log-return distribution. A third difference is that the distribution from market closings is usually leptokurtic since the coefficient of kurtosis [2], $\eta_4 \equiv M_4/(M_2)^2 > 3$, where the normal distribution kurtosis value is 3 and M_4 is the fourth central moment. This means that the tails are fatter than a normal distribution with the same mean and standard deviation, so that the distribution is also more slender about the mode (local maximum) to conserve probability. A fourth difference is that the market exhibits time-dependence in the distributions of log-returns, so that the associated parameters are time-dependent.

Merton [12, Chap. 9] introduced Poisson jumps with independent identically distributed random jump-amplitudes with fixed mean and variances into the Black-Scholes model, but the ability to hedge the volatilities was not very satisfactory. Kou [11] uses a jump-diffusion model with a double exponential jump-amplitude distribution with mean κ and variance 2η, having leptokurtic and negative skewness properties and has many analytical advantages, although it is difficult to see the empirical justification for this distribution.

Hanson and Westman [5] reformulated an important external events model of Rishel [13] solely in terms of stochastic differential equations and applied it to the computation of the optimal portfolio and consumption policies problem for a portfolio of stocks and a bond. The stock prices depend on both scheduled and unscheduled jump external events. The computations were illustrated with a simple log-bi-discrete (i.e., 2 discrete jumps in the log-return, one negative and one positive) jump-amplitude model, either negative or positive jumps, such that both stochastic and quasi-deterministic jump magnitudes were estimated. In [7], they constructed a jump-diffusion model with marked Poisson jumps that had a log-normally distributed jump-amplitude and rigorously derived the density function for

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the log-normal-diffusion and log-normal-jump stock price log-return model. In [6], this financial model is applied to the optimal portfolio and consumption problem for a portfolio of stocks and bonds including computational results. In [9], they treated the case of time-dependent parameters with uniformly distributed jump amplitudes estimating the jump intensity using an unweighted least squares method with constraints on the other jump diffusion parameters. In [8], they derived a proper weighting from jump-diffusion theory for a weighted least squares that emphasized the jump component of the distribution more, thus yielding better fits to the jump parameters. This paper presents an improved fitting using a weighted least squares to estimate the jump intensity and the jump amplitude interval.

In particular, this paper treats the log-normal-diffusion, log-uniform-jump problem. In Section 2, the derivation of the jump-diffusion density is briefly discussed using a modification of our prior theorem [7]. In Section 3, the time dependent parameters for this log-return process are estimated using this theoretical density and the S&P500 Index daily closing data for the prior decade. Concluding remarks are given in Section 4.

2. Log-Return Density for Log-Normal-Diffusion, Log-Uniform Jump

Let $S(t)$ be the price of a stock or mutual fund, that is governed by a geometric jump-diffusion stochastic differential equation (SDE) with time-dependent coefficients,

$$dS(t) = S(t)\left[\mu_d(t)dt + \sigma_d(t)dZ(t) + J(t)dP(t)\right] , \quad (1)$$

with $S(0) = S_0, \;\; S(t) > 0$, where $\mu_d(t)$ is the diffusive drift at time t, $\sigma_d(t)$ is the diffusive volatility, $Z(t)$ is is a continuous, one-dimensional Gaussian process, $P(t)$ is a discontinuous, one-dimensional standard Poisson process with jump rate $\lambda(t)$, and associated jump-amplitude $J(t)$ with log-return mean $\mu_j(t)$ and variance $\sigma_j^2(t)$. The stochastic processes $Z(t)$ and $P(t)$ are assumed to be Markov and pairwise independent. The jump-amplitude $J(t)$, given that a Poisson jump in time occurs, is also independently distributed. The stock price SDE (1) is similar in our prior work [7, 6], except that time-dependent coefficients introduce more realism as in [9].

The continuous, differential diffusion process $dZ(t)$ is the standard Gaussian process with zero mean and dt variance. The symbolic notation for the discontinuous space-time jump process, $J(t)dP(t)$, is better defined in terms of the Poisson random measure, $\mathcal{P}(dt, dq)$, by the stochastic integral, $J(t)dP(t) = \int_{\mathcal{Q}} \widehat{J}(t;q)\mathcal{P}(dt,dq)$, where $Q = q$ is the Poisson spatial mark variable for the jump amplitude process, and $\widehat{J}(t;q)$ is the kernel of the Poisson operator $J(t)$, such that $-1 < \widehat{J}(t;q) < \infty$ so that a single jump does not make the underlying non-positive. The infinitesimal moments of the jump process are $\mathrm{E}[J(t)dP(t)] = \lambda(t)dt \int_{\mathcal{Q}} \widehat{J}(t;q)\phi_Q(q;t)dq$ and $\mathrm{Var}[J(t)dP(t)] = \lambda(t)dt \int_{\mathcal{Q}} \widehat{J}^2(t;q)\phi_Q(q;t)dq$, neglecting $O^2(dt)$ here, where $\phi_Q(q;t)$ is the Poisson amplitude mark density. The differential Poisson process is a counting process with the probability of the jump count given by the usual Poisson distribution, $p_k(\lambda(t)dt) = \exp(-\lambda(t)dt)(\lambda(t)dt)^k/k!$, $k = 0, 1, 2, \ldots$, with parameter $\lambda(t)dt > 0$. For a clear, more rigorous presentation of jump-diffusion theory see Runggaldier's [14] handbook chapter.

Since the stock price process is geometric, the common multiplicative factor of $S(t)$ can be transformed away yielding the SDE of the stock price log-return with state-independent right hand side using the stochastic chain rule for Markov processes in continuous time,

$$\begin{aligned} d[\ln(S(t))] &= \mu_{ld}(t)dt + \sigma_d(t)dZ(t) \\ &\quad + \ln(1 + J(t))dP(t), \end{aligned} \quad (2)$$

where $\mu_{ld}(t) \equiv \mu_d(t) - \sigma_d^2(t)/2$ is the log-diffusion (*ld*) drift and $\ln(1 + \widehat{J}(t;q))$ is the stock log-return jump-amplitude, the logarithm of the relative post-jump-amplitude. This log-return SDE (2) will be the model that will used for comparison to the S&P500 log-returns. Since $\widehat{J}(t;q) > -1$, it is convenient to select the mark process to be the jump-amplitude random variable, $Q = \ln\left(1 + \widehat{J}(t;Q)\right)$, on the mark space $\mathcal{Q} = (-\infty, +\infty)$. Though this is a convenient mark selection, it implies the time-independence of the jump-amplitude, so $\widehat{J}(t;Q) = \widehat{J}_0(Q)$ or $J(t) = J_0$. Since market jumps are rare and the tails are relatively flat, a reasonable approximation is a uniform jump-amplitude distribution with density $\phi_Q(q;t)$ on the finite, time-dependent mark interval $[Q_a(t), Q_b(t)]$,

$$\phi_Q(q;t) \equiv \frac{H(Q_b(t) - q) - H(Q_a(t) - q)}{Q_b(t) - Q_a(t)}, \quad (3)$$

where $H(x)$ is the Heaviside, unit step function. The density $\phi_Q(q;t)$ yields a mean $\mathrm{E}_Q[Q] = \mu_j(t) = (Q_b(t) + Q_a(t))/2$ and variance $\mathrm{Var}_Q[Q] = \sigma_j^2(t) = (Q_b(t) - Q_a(t))^2/12$, which define the basic log-return jump amplitude moments. It is assumed that $Q_a(t) < 0 < Q_b(t)$, to make sure that both negative and positive jumps are represented, which was a problem for the log-normal jump-amplitude distribution in [6]. The uniform distribution is treated as time-dependent in this paper, so $Q_a(t)$, $Q_b(t)$, $\mu_j(t)$ and $\sigma_j^2(t)$ all depend on t.

The difficulty in separating out the small jumps about the mode or maximum of real market distributions is explained by the fact that a diffusion approximation for small marks can be used for the jump process that will be indistinguishable from the continuous Gaussian process anyway. Thus, there does not seem that there can be a theoretical justification for use of the double exponential jump distribution [11] with peak in the diffusion part of the distribution, except for the convenience of extensive analysis. The use of the simple uniform jump distribution is quite sufficient since the jumps are most clearly detectable as outliers in the tail of the financial market distribution.

The basic moments of the stock log-return increment ($dt \to \Delta t$) are

$$M_1^{(\mathrm{jd})} \equiv \mathrm{E}[\Delta[\ln(S(t))]] = (\mu_{ld}(t) + \lambda(t)\mu_j(t))\Delta t, \tag{4}$$

$$\begin{aligned} M_2^{(\mathrm{jd})} &\equiv \mathrm{Var}[\Delta[\ln(S(t))]] \\ &\simeq \left(\sigma_d^2(t) + \lambda(t)\left(\sigma_j^2(t)\mu_j^2(t)\right)\right)\Delta t, \end{aligned} \tag{5}$$

where the $O^2(\Delta t)$ term has been neglected in the variance, since the discrete return time, $dt = \Delta t$, the daily fraction of one trading year (about 250 days), will be small.

The log-normal-diffusion, log-uniform-jump density can be found by basic probabilistic methods following a slight modification for time-dependent coefficients of constant coefficient theorem found our paper [7],

Theorem 2.1:
The probability density for the log-normal-diffusion, log-uniform-jump amplitude log-return increment $\Delta[\ln(S(t))]$ specified in the SDE (2) with time-dependent coefficients is given by

$$\begin{aligned} \phi^{(\mathrm{jdth})}(x) &\sim p_0(\lambda(t)\Delta t)\phi^{(n)}\left(x; \mu_{ld}(t)\Delta t, \sigma_d^2(t)\Delta t\right) \\ &+ \tfrac{p_1(\lambda(t)\Delta t)}{(Q_b(t)-Q_a(t))} \\ &\cdot \Big[\Phi^{(n)}\left(Q_b(t) - x + \mu_{ld}(t)\Delta t; 0, \sigma_d^2(t)\Delta t\right) \\ &- \Phi^{(n)}\left(Q_a(t) - x + \mu_{ld}(t)\Delta t; 0, \sigma_d^2(t)\Delta t\right)\Big] , \end{aligned} \tag{6}$$

for sufficiently small Δt and $-\infty < x < +\infty$, where $p_k(\lambda(t)\Delta t)$ is the Poisson distribution and the normal distribution with mean $\mu_{ld}\Delta t$ and variance $\sigma_d^2 \Delta t$ is

$$\Phi^{(n)}(x; \mu_{ld}\Delta t, \sigma_d^2\Delta t) = \int_{-\infty}^{x} \phi^{(n)}(y; \mu_{ld}\Delta t, \sigma_d^2\Delta t)dy$$

associated with $d\ln(S(t))$, the diffusion part of the log-return process,

$$\phi_{\mu_{ld}\Delta t + \sigma_d dZ(t)}(x) = \phi^{(n)}(x; \mu_{ld}(t)\Delta t, \sigma_d^2(t)\Delta t) .$$

The proof, which is only briefly sketched here, follows from the density of a triad of independent random variables, $\xi + \eta \cdot \zeta$ given the densities of the three component processes ξ, η, and ζ. Here, (1) $\xi = \mu_{ld}(t)\Delta t + \sigma_d(t)\Delta Z(t)$ is the log-normal plus log-drift diffusion process, (2) $\eta = Q = \ln(1 + \widehat{J}_0(Q))$ is the log-uniform jump-amplitude, and (3) $\zeta = \Delta P(t)$ is the differential Poisson process. The density of a sum of independent random variables, as in the sum operation of $\xi + (\eta \cdot \zeta)$, is very well-known and is given by a convolution of densities $\phi_{\xi+\eta\zeta}(z) = \int_{-\infty}^{+\infty} \phi_\xi(z-y)\phi_{\eta\zeta}(y)dy$ (see Feller [3]). However, the distribution of the product of two random variables $\eta \cdot \zeta$ is not so well-known [7] and has the density,

$$\begin{aligned} \phi_{\eta\zeta}(x) &\sim p_0(\lambda(t)\Delta t)\delta(x) \\ &+ \tfrac{p_1(\lambda(t)\Delta t)[H(Q_b(t)-x)-H(Q_a(t)-x)]}{(Q_b(t)-Q_a(t))} , \end{aligned} \tag{7}$$

for the log-uniform-jump process and sufficiently small Δt. The probabilistic mass at $x = 0$ is represented by the Dirac $\delta(x)$ and corresponds to the zero jump event case. Finally, applying the convolution formula for density of the sum $\xi + (\eta\zeta)$ leads to the density for the jump-diffusion random variable triad $\xi + \eta\zeta$ given asymptotically in (6) of the theorem.

3. Jump-Diffusion Parameter Estimation

Given the log-normal-diffusion, log-uniform-jump density (6), it is necessary to fit this theoretical model to realistic empirical data to estimate the parameters of the log-return model (2) for $d[\ln(S(t))]$. For realistic empirical data, the daily closings of the S&P500 Index during the decade from 1992 to 2001 are used from data available on-line [15]. The data consists of $n^{(\mathrm{sp})} = 2522$ daily closings. The S&P500 data can be viewed as an example of one large mutual fund rather than a single stock. The data has been transformed into the discrete analog of the continuous log-return, i.e., into changes in the natural logarithm of the index closings, $\Delta[\ln(SP_i)] \equiv \ln(SP_{i+1}) - \ln(SP_i)$ for $i = 1, \ldots, n^{(\mathrm{sp})} - 1$ daily closing pairs. For the decade, the mean is $M_1^{(\mathrm{sp})} \simeq 4.015 \times 10^{-4}$ and the variance is $M_2^{(\mathrm{sp})} \simeq 9.874 \times 10^{-5}$, the coefficient of skewness is $\eta_3^{(\mathrm{sp})} \equiv M_3^{(\mathrm{sp})}/(M_2^{(\mathrm{sp})})^{1.5} \simeq -0.2913 < 0$, demonstrating the typical negative skewness property, and the coefficient of kurtosis is $\eta_4^{(\mathrm{sp})} \equiv M_4^{(\mathrm{sp})}/(M_2^{(\mathrm{sp})})^2 \simeq 7.804 > 3$, demonstrating the typical leptokurtic behavior of many real markets.

The S&P500 log-returns, $\Delta[\ln(SP_i)]$ for $i = 1 : n^{(\mathrm{sp})}$ decade data points, are partitioned into 10 yearly data sets, $\Delta[\ln(SP_{j_y,k}^{(\mathrm{spy})})]$ for $k = 1 : n_{y,j_y}^{(\mathrm{sp})}$ yearly data points for $j_y = 1 : 10$ years, where $\sum_{j_y=1}^{10} n_{y,j_y}^{(\mathrm{sp})} = n^{(\mathrm{sp})}$. For each of these yearly sets, the parameter estimation objective is to find the least sum of weighted squares of the deviation between the empirical S&P500 log-return histograms for the year and the analogous theoretical log-normal-diffusion, log-uniform-jump distribution histogram based upon the same bin structure of 100 bins for each year. In particular, the weighted formulation is given as

$$\chi^2 = \sum_{i=1}^{N^{(\mathrm{bin})}} \omega_i \cdot \left(f_i^{(\mathrm{jdth})} - f_i^{(\mathrm{sp})}\right)^2 , \tag{8}$$

where ω_i is the weight of the ith bin, $f_i^{(\mathrm{sp})}$ is the ith empirical S&P500 bin frequency data and $f_i^{(\mathrm{jdth})}$ is the ith theoretical jump-diffusion bin frequency corresponding to the same sample size $N^{(\mathrm{sp})} = 2521$.

In [8], we derived the proper weights for this least squares procedure from the jump-diffusion distribution and the theory by which the distribution is approximated by jump-diffusion simulations. This result is summarized in the following:

Theorem 3.1:
If $f_i^{(\mathrm{jdsim})} = \sum_{j=1}^{N} U(\Delta S_j^{(\mathrm{jdsim})}; x_i, x_{i+1}^-)$ for $i = 1 : N^{(\mathrm{bin})}$ are the frequencies of the ith bin $[x_i, x_{i+1})$ and $\Delta S_j^{(\mathrm{jdsim})}$ is the jth jump-diffusion simulation, using N samples, as prescribed for (2), then the bin frequency ex-

pectation and variance are

$$
\begin{aligned}
\mu_{f_i^{(\mathrm{jdsim})}} &= \mathrm{E}\left[f_i^{(\mathrm{jdsim})}\right] = f_i^{(\mathrm{jdth})} \\
\sigma^2_{f_i^{(\mathrm{jdsim})}} &= \mathrm{Var}\left[f_i^{(\mathrm{jdsim})}\right] \\
&= N \cdot \left(1 - \tfrac{1}{N} f_i^{(\mathrm{jdth})}\right)^2 f_i^{(\mathrm{jdth})},
\end{aligned}
\tag{9}
$$

respectively, where the i*th expected bin frequency after* N *simulations is*

$$
f_i^{(\mathrm{jdth})} = N \cdot \int_{x_i}^{x_{i+1}} \phi_i^{(\mathrm{jdth})}(x)dx.
$$

Here, $U(x;a,b)$ *is the unit step function on* (a,b)*.*

The bin weights are chosen as the normalized theoretical values,

$$
\omega_i = \omega_i^{(\mathrm{jdth})} = \left(1/\sigma^2_{(f_i^{(\mathrm{jdth})})}\right) \Bigg/ \sum_{j=1}^{N^{(\mathrm{bin})}} \left(1/\sigma^2_{(f_j^{(\mathrm{jdth})})}\right) \ , \tag{10}
$$

for $i = 1 : N^{(\mathrm{bin})}$ bins, normalized to a unit sum for convenience of small minima.

Keeping the number of free parameters as small as practical, we require that the mean and variances of the yearly log-returns be the same for both empirical and theoretical distributions, i.e., our moment constraints are

$$
M_{1,j_y}^{(\mathrm{spy})} \equiv \mathrm{Mean}_{k=1}^{n_{y,j_y}^{(\mathrm{sp})}} \left[\Delta\left[\ln\left(SP_{j_y,k}^{(\mathrm{spy})}\right)\right]\right] = M_{1,j_y}^{(\mathrm{jdy})} \tag{11}
$$

using (4) and

$$
M_{2,j_y}^{(\mathrm{spy})} \equiv \mathrm{Var}_{k=1}^{n_{y,j_y}^{(\mathrm{sp})}} \left[\Delta\left[\ln\left(SP_{j_y,k}^{(\mathrm{spy})}\right)\right]\right] = M_{2,j_y}^{(\mathrm{jdy})} \tag{12}
$$

using (5), for each $j_y = 1 : 10$ years. Since the primary interest here is the jump component of the process, the constraints (11,12 are chosen to constrain the log-diffusion parameters, such that

$$
\mu_{ld,j_y} = \left(M_{1,j_y}^{(\mathrm{spy})} - (\lambda\ dt\mu_j)_{j_y}\right)/\Delta T_{j_y}\ , \tag{13}
$$

$$
\sigma^2_{d,j_y} = \left(M_{2,j_y}^{(\mathrm{spy})} - (\lambda dt((1+\lambda dt)\sigma_j^2 + \mu_j^2)_{j_y})\right)/\Delta T_{j_y}\ , \tag{14}
$$

with $\sigma^2_{d,j_y} > 0$ for each $j_y = 1 : 10$ years. Of the six parameters $\{\mu_{ld,j_y}, \sigma^2_{d,j_y}, \mu_{j,j_y}, \sigma^2_{j,j_y}, \lambda_{j_y}, \Delta T_{j_y}\}$, needed for each year j_y to specify the jump-diffusion log-return distribution, only the three jump parameters $\{Q_a, Q_b, \lambda\Delta T\}_{j_y}$ needs to be estimated by a three dimensional nonlinear least squares, using the first two moments for constraints. The time step $\Delta t = \Delta T_{j_y}$ is given as the reciprocal of the number of trading days per year, close to 250 days, but varies a little for $j_y = 1 : 10$ and has values lying in the range, [0.003936, 0.004050], of small values and are used here for parameter estimation.

Thus, we have a three dimensional global minimization problem for a highly complex discretized jump-diffusion density function (6). The analytical complexity indicates that a general global optimization method that does not require derivatives would be useful. For this purpose, such a method, *Golden Super Finder (GSF)* [10], was developed for [6, 9, 8] and implemented in MATLAB™ , since simple techniques are desirable in financial engineering. The *GSF* method is an extensive modification to the Golden Section Search method [4], extended to multi-dimensions and allowing search beyond the initial hyper-cube domain by including the endpoints in the local optimization test with the two golden section interior points per dimension, moving rather than shrinking the hypercube when the local optimum is at an edge or corner. The method, as a general method, is slow, but systematically moves the search until the uni-modal optimum is found at a interior point and then approaches the optimum if within the original search bounds. Additional constraints can be added to the objective function, such as (13,14). If the diffusion coefficient vanishes, $\sigma_d^2 \to 0^+$, then (14) implies a maximum jump count constraint,

$$
\begin{aligned}
\max[\lambda \cdot \Delta t] \quad = \quad & 0.5(\sqrt{((\sigma_j^2 + \mu_j^2)^2 + 4\sigma_j^2 \cdot M_2)} \\
& -(\sigma_j^2 + \mu_j^2))/\sigma_j^2.
\end{aligned}
$$

An additional compatibility constraint, $\sigma_j(t) > 0$, does not need enforcement as long as $Q_a(t) < Q_b(t)$ and is not violated here.

In the next three figures, a a sample comparison can be made of the empirical S&P500 histogram on the left of each yearly figure with the corresponding theoretical jump-diffusion histogram on the right. The jump-diffusion histogram is a very idealized version of the empirical distribution, with the asymmetry of the tails clearly illustrated, noting that the years 1997-present are more noisier than the quieter years from 1992-1995. Figure 1 for 1993 represents a quieter year, Figure 2 for 1996 represents an intermediate year, and Figure 3 for 2000 represents a noisier year, The histograms for the yearly empirical data on the left sides suggest that it may take more than a year of 250 or so closings to develop a more typical market log-return distribution. For interpreting these results one needs to keep in mind what a moderate number of simulations of an underlying distribution looks like and the yearly sample of about 250 closings is not very large.

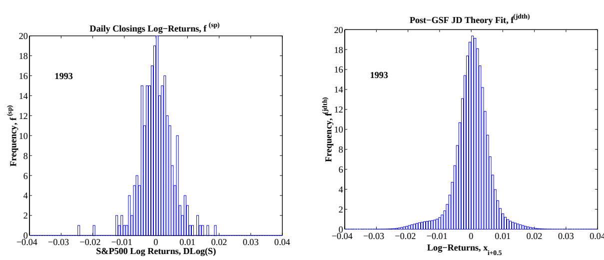

Figure 1: Comparison for the relatively quiet year 1993 of the empirical S&P500 histogram on the left with the corresponding fitted theoretical jump-diffusion histogram on the right, using 100 bins.

The fitting procedure is a highly nonlinear application of least squares, since the fitted jump-diffusion distribution and the weights on dependent on the parameters in a highly

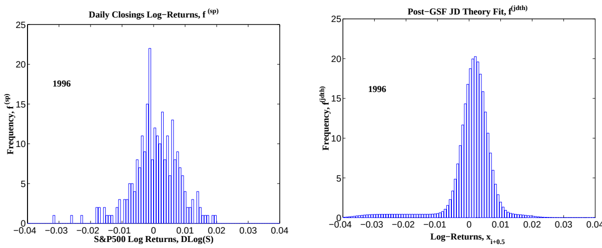

Figure 2: Comparison for the intermediate year 1996 of the empirical S&P500 histogram on the left with the corresponding fitted theoretical jump-diffusion histogram on the right, using 100 bins.

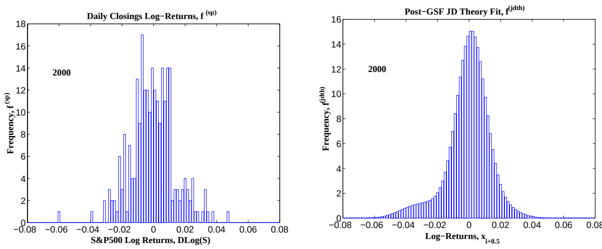

Figure 3: Comparison for the intermediate year 1996 of the empirical S&P500 histogram on the left with the corresponding fitted theoretical jump-diffusion histogram on the right, using 100 bins.

nonlinear way. The computation is also highly computationally intensive even though only about 12 iterations were needed for a hybrid value-position relative change converge to a tolerance of 5.e-3. The three dimensional free parameter space search required $4^3 = 64$ function evaluations of χ^2 per iteration, taking about 4 hours per year on a 400MHz Pentium II. The weights ω_i in (10) also significantly participate in the highly nonlinear least squares parameter optimization. In Fig. 4, the histograms of the bin frequency standard deviations, $\sigma_{f_i^{(\mathrm{jdth})}}$, proportional to the reciprocal square root of the weights, are exhibited for the quieter year 1993 and the noisier year 2000. The bin

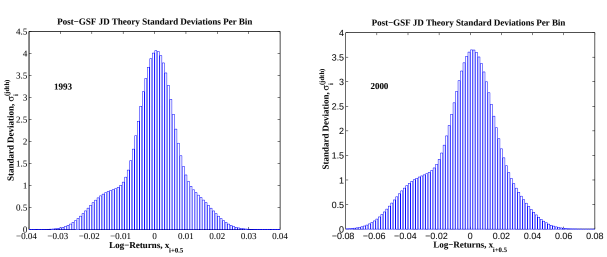

Figure 4: Optimal jump-diffusion bin standard deviations $\sigma_{f_i^{(\mathrm{jdth})}}$ for the relatively quiet year 1993 on the left and the relatively noisier year 2000 on the right.

standard deviations resemble the theoretical jump-diffusion histograms "*on steroids*", in that the tails are more exaggerated, giving more weight and thus more importance to the outlying jumps. Note that the actual parameters are more than the size of the hybrid relative tolerance would indicate, since parameters during the final interval undergo much smaller changes.

The final free parameters fit in this highly nonlinear least squares are listed in Table 1. Note that the uniform inter-

Table 1: Summary of estimated yearly free parameters for Log-Normal-Diffusion, Log-Uniform-Jump distribution by weighted least squares based on the deviation between S&P500 and jump-diffusion histograms, with respect to the free parameter set $\{Q_a, Q_b, \lambda\Delta t\}T_{j_y}$ given $\Delta t = \Delta T_{j_y}$ and constraints cited in the text.

Year$_{j_y}$	Q_{a,j_y}	Q_{b,j_y}	$(\lambda\Delta T)_{j_y}$
1992	-0.01418	0.01373	0.2549
1993	-0.01957	0.01518	0.1471
1994	-0.01425	0.01430	0.2891
1995	-0.01948	0.01749	0.2118
1996	-0.03765	0.01794	0.1185
1997	-0.05310	0.02364	0.1671
1998	-0.06138	0.03627	0.1147
1999	-0.01957	0.04116	0.2841
2000	-0.04503	0.02732	0.2287
2001	-0.05109	0.03177	0.1682

vals Q_{a,j_y}, Q_{b,j_y} are much smaller during the quieter period, 1992-1995, than in the noisier period 1997-2001, excluding for the exceptional year of 1999. In [9], difficulty was found trying to fit the jump-diffusion using the one parameter fit with only $(\lambda\Delta T)_{j_y}$ free due to requiring many more iterations and finding a negligible jump rate, whereas the three parameter values for 1999 are not that different in the extreme.

Using these approximately optimized free jump parameters, the values of the yearly jump-diffusion parameters can be derived, using the two moment constraints and the yearly time step, with the diffusion means and variances given in Table 2, while the jump means and variances are given in Table 3. Viewing Table 3, the nearly zero diffusive

Table 2: Summary of derived yearly diffusion distribution parameters for Log-Normal-Diffusion, Log-Uniform-Jump distribution by weighted least squares.

Year$_{j_y}$	μ_{d,j_y}	σ_{d,j_y}
1992	+0.05957	0.07211
1993	+0.1502	0.05900
1994	-0.01319	0.06867
1995	+0.3545	2.366e-7?
1996	+0.04700	0.05728
1997	+0.9040	0.05568
1998	+0.6005	0.1170
1999	-0.6022	0.04690
2000	+0.3949	0.1423
2001	+0.2987	0.1308

volatility σ_{d,j_y} for 1995 puts it out of place in the quieter period 1992-1995, while the diffusive volatility for the year 1999 is about half of that of the other years on the noisier period of 1997-2001. These peculiarities bear further investigation. The typical jump rate is about 50 per year, except for the the years 1997-1998 where it is almost half

Table 3: Summary of derived yearly jump distribution parameters for Log-Normal-Diffusion, Log-Uniform-Jump distribution by weighted least squares.

Year_{j_y}	μ_{j,j_y}	σ_{j,j_y}	λ_{j_y}
1992	-2.218e-4	8.058e-3	53.34
1993	-2.196e-3	3.271e-3	45.62
1994	+2.546e-5	8.242e-3	52.92
1995	-9.968e-4	1.067e-2	52.92
1996	-9.854e-3	1.605e-2	53.34
1997	-1.473e-2	2.215e-2	24.76
1998	-1.256e-2	2.819e-2	31.46
1999	+1.080e-2	1.753e-2	52.92
2000	-8.858e-3	2.089e-2	50.29
2001	-9.662e-3	2.392e-2	51.87

of that. This typical high rate, though, counts those jumps in the central diffusive region of the distribution since the uniform jump distribution spans that region. A bi-normal uniform distribution could be used that skips most of the central diffusive region, but then 2 more parameters would be needed to cut off the central part of the positive jump part and the negative jump part of the bi-uniform distribution, plus still a third additional parameter that would be needed to represent the relative probability of positive or negative jumps.

The summaries of the coefficients of skewness and kurtosis are given in Table 4 for the empirical S&P500 data and the estimated theoretical jump-diffusion distribution for comparison. Note that the theoretical jump-diffusion values seem to be better in the noisier period than in the quieter period, with the years 1995 and 1999 being especially different. There are obvious numerical difficulties in trying to reproduce statistical moments as high as third and fourth order, due to limitations on well-conditioning the calculation against catastrophic cancellation.

Table 4: Summary of yearly coefficients of skewness, η_3, and kurtosis less the normal value, $\eta_4 - 3$, for both the estimated theoretical jump–diffusion (superscript $(jdth)$) model and empirical S&P500 (superscript (sp)) decade data.

Year_{j_y}	$\eta^{(\text{jd})}_{3,j_y}$	$\eta^{(\text{sp})}_{3,j_y}$	$\eta^{(\text{jd})}_{4,j_y} - 3$	$\eta^{(\text{sp})}_{4,j_y} - 3$
1992	-3.6e-1	+5.9e-2	8.1e-1	2.4e-1
1993	-5.3e-1	-1.8e-1	2.7	2.4
1994	+4.5e-3	-3.0e-1	8.4e-1	1.3
1995	-8.6e-1	-8.1e-2	3.6e+1	1.0
1996	-2.2	-6.0e-1	8.0	1.7
1997	-2.4	-6.9e-1	7.2	6.5
1998	-1.5	-6.2e-1	6.3	4.7
1999	+1.6	+6.8e-2	3.0	-1.5e-1
2000	-7.8e-1	-1.4e-2	1.8	1.3
2001	-1.0	+3.1e-2	3.2	1.5

Further, the concept that the market data is usually negatively skewed does not hold for periods as short as one trading year, since 1999 has positive skew in both empirical data and jump diffusion model, while 1992 exhibited positive skew from the data. Also, the concept that the market data is usually leptokurtic apparently refers to long term data and not to shorter term data, since the 1999 kurtosis coefficient less the normal value is negative, though small in magnitude.

4. Conclusions

The main contributions of this paper are the introduction of the uniformly distributed jump-amplitude into the jump-diffusion stock price model model and the fitting of time-dependent in the jump-diffusion parameters using a novel combination of jump-diffusion motivated by a highly non-linear weighted least squares. The uniformly distributed jump-amplitude feature of the model is a reasonable assumption for rare, large jumps when there is only a sparse population of isolated jumps in the tails of the market distribution. The jump-diffusion weighted least squares method gives greater emphasis to the jumps than the non-weighted (unit weights) version of least squares. While in many statistical analyses, the outlier are discarded, but in finance they represent important market events. The jumps, whether crashes or buying frenzies, represent important events in the market in the background of continuous diffusive noise. Additional realism in the jump-diffusion model is given by the introduction of time dependence in the distribution and in the associated parameters.

Further improvements, but with greater computational complexity, would be to estimate the uniform distribution limits $[Q_a, Q_b]$ by fitting the theoretical distribution to real market distributions, using longer and overlapping partitioning of the market data to reduce the effects of small sample sizes. Other future directions, is to apply the results to optimal portfolio computations and approximate hedging.

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