The LQGP Problem: A Manufacturing Application[1]

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Abstract

The Linear Quadratic Gaussian Poisson (LQGP) problem denotes an optimal control problem with linear dynamics and quadratic costs with both Gaussian and Poisson noise disturbances. The LQGP problem provides a benchmark model with sufficient complexity while permitting formal solutions for testing both theoretical and computational methods. The problem is examined and is illustrated with a flexible, multistage manufacturing system application.

1. Introduction

The linear dynamics, quadratic performance, Gaussian noise and Poisson noise or LQGP problem, has its dynamics governed by the stochastic differential equation (SDE)

$$\begin{aligned} d\mathbf{X}(t) &= [A(t)\mathbf{X}(t) + B(t)\mathbf{U}(t) + \mathbf{C}(t)]dt \\ &+ G(t)d\mathbf{W}(t) + [H_1(t)\mathbf{X}(t)]d\mathbf{P}_1(t) \\ &+ [H_2(t)\mathbf{U}(t)]d\mathbf{P}_2(t) + H_3(t)d\mathbf{P}_3(t), \end{aligned} \tag{1}$$

for general Markov processes in continuous time, with $m \times 1$ state vector **X**(t), $n \times 1$ control vector **U**(t), $r \times 1$ Gaussian noise vector d**W**(t), and $q_\ell \times 1$ space-time Poisson noise vectors $d\mathbf{P}_\ell(t)$, for $\ell = 1$ to 3. The dimensions of the respective coefficient matrices are: $A(t)$ is $m \times m$, $B(t)$ is $m \times n$, $\mathbf{C}(t)$ is $m \times 1$, $G(t)$ is $m \times r$, while the $H_\ell(t)$ are dimensioned, so that $[H_1(t)\mathbf{x}] = [\sum_k H_{1ijk}(t)x_k]_{m\times q_1}$, $[H_2(t)\mathbf{u}] = [\sum_k H_{2ijk}(t)u_k]_{m\times q_2}$ and $H_3(t) = [H_{3ij}(t)]_{m\times q_3}$. Note that the space-time Poisson terms are formulated to maintain the linear nature of the dynamics, but the first two are actually bilinear in either **X** or **U** and $d\mathbf{P}_\ell$ for $\ell = 1$ or 2, respectively. This is necessary so that a modification of the LQG analysis (see Bryson and Ho [5] or Lewis [11]) will work from the dynamic programming point of view. It is assumed that the SDE (1) is interpreted in the sense of Itô.

The LQGP problem with linear dynamics in the form of (1) has its origins in an early set of lecture notes by Wonham [13] on random differential equations. A similar formulation, but including jumps in the system parameters along with Poisson disturbances such as in (1), is discussed by Mariton [12] in his monograph on optimal control of linear systems with random jumps in system parameters. The latter problem with quadratic costs is more often is referred to as the *Jump LQG* or *JLQG* problem. The LQGP problem could also be considered as a JLQG problem, since Poisson term introduces a discrete jump in the state, superimposed on the continuous, yet nonsmooth Gaussian noise contribution. However, the JLQG problem covers noise other than Poisson and there is quite a difference between the Markov chain noise introduced through system parameters and disturbances introduced as a term in the dynamics as in (1). The more general and current name for combinations of discrete (e.g., Markov chains or Poisson noise) and continuous systems is *hybrid systems*.

The machinery for these generalizations is found in Hanson and Ryan [9] and related papers, where a concrete, constructive proof is given for the equivalency between the state mean and the quasi-deterministic mean derived form the infinitesimal mean and variance for general continuous time Markov noise with linear amplitudes.

In Section 2, the moments of the SDE (1) are derived and the quadratic performance index is described in Section 2.1. In Section 3, the dynamic programming formulation is used to derive the formal optimal solution in Section 3.1. In Section 4, a computational example for a multistage manufacturing system is considered.

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2. Infinitesimal Conditional Moments

The Gaussian white noise term, $d\mathbf{W}(t)$, consists of r independent, standard Wiener processes $dW_i(t)$, for $i = 1$ to r. These Gaussian components have zero infinitesimal mean and diagonal covariance. We have left the Gaussian noise coefficient independent of both state and control vectors for simplicity, but $G(t)$ could well be generalized like those of the Poisson noise coefficients, $[H_1(t)\mathbf{x}]$ and $[H_2(t)\mathbf{u}]$.

The space-time Poisson noise terms, $d\mathbf{P}_\ell(t) = [dP_{\ell,i}(t)]_{q_\ell \times 1}$, consist of q_ℓ independent differentials of space-time Poisson processes each. They are related to Poisson random measure, $\mathcal{P}_{\ell,i}(dz_i, dt)$, formulation (see Gihman and Skorohod [7]): $dP_{\ell,i}(t) = \int_{\mathcal{Z}_{\ell,i}} z_i \mathcal{P}_{\ell,i}(dz_i, dt)$, where z_i is the Poisson amplitude random variable of the $dP_{\ell,i}(t)$ processes, for $\ell = 1$ to 3 while $i = 1$ to q_ℓ. These have mean

$$\text{Mean}[d\mathbf{P}_\ell(t)] = \Lambda_\ell(t)dt \left[\int_{\mathcal{Z}_{\ell,i}} z_i \phi_{\ell,i} dz_i\right]_{q_\ell \times 1} \equiv \Lambda_\ell \overline{\mathbf{Z}}_\ell dt, \quad (2)$$

where $\Lambda_\ell(t)$ is the diagonal representation of the Poisson rates $\lambda_{\ell,i}(t)$, $\overline{\mathbf{Z}}_\ell(t)$ is the vector mean and $\phi_{\ell,i}(z_i, t)$ is the density of the $\{\ell, i\}$th amplitude mark component, and covariance

$$\begin{aligned} \text{Covar}[d\mathbf{P}_\ell(t)] &= \Lambda_\ell(t)dt \left[\int_{\mathcal{Z}_{\ell,i}} (z_i - \overline{Z}_{\ell,i})^2 \phi_{\ell,i} dz_i\right]_{q_\ell \times 1} \quad (3) \\ &\equiv \Lambda_\ell(t)\sigma_\ell(t)dt = [\lambda_{\ell,i}\sigma_{\ell,i}\delta_{i,j}]_{q_\ell \times 1} \end{aligned}$$

with $\sigma_\ell(t)$ denoting the diagonalized variance of the mark distribution for $\mathbf{P}_\ell(t)$. It is further assumed that all of the individual component terms of the Gaussian and Poisson noises are independent, i.e., $\text{Covar}[d\mathbf{W}(t), d\mathbf{P}_\ell^T(t)] = 0_{r \times q_\ell}$.

The jth jump of the $\{\ell, i\}$th space-time Poisson process at time $t_{\ell,i,j}$ with amplitude $z_{\ell,i,j}$ causes the following jump from $t^-_{\ell,i,j}$ to $t^+_{\ell,i,j}$ in the state:

$$[\mathbf{X}](t_{\ell,i,j}) = \left\{ \begin{array}{ll} [H_1(t_{\ell,i,j})\mathbf{X}(t^-_{\ell,i,j})]_i z_{\ell,i,j}, & \ell = 1 \\ [H_2(t_{\ell,i,j})\mathbf{U}(t^-_{\ell,i,j})]_i z_{\ell,i,j}, & \ell = 2 \\ [H_3(t_{\ell,i,j})]_i z_{\ell,i,j}, & \ell = 3 \end{array} \right\}. \quad (4)$$

From the above statistical properties of the stochastic processes, $d\mathbf{W}$ and $d\mathbf{P}_\ell$, it follows that the conditional infinitesimal expectation of the state is

$$\begin{aligned} &\text{Mean}[d\mathbf{X}(t) \mid \mathbf{X}(t) = \mathbf{x}, \mathbf{U}(t) = \mathbf{u}] \quad (5) \\ &= \Big[A(t)\mathbf{x} + B(t)\mathbf{u} + \mathbf{C}(t) + [H_1(t)\mathbf{x}]\Lambda_1\overline{\mathbf{Z}}_1(t) \\ &+ [H_2(t)\mathbf{u}]\Lambda_2\overline{\mathbf{Z}}_2(t) + H_3(t)\Lambda_3\overline{\mathbf{Z}}_3(t)\Big]\, dt, \end{aligned}$$

and the conditional infinitesimal covariance,

$$\begin{aligned} &\text{Covar}[d\mathbf{X}(t) \mid \mathbf{X}(t) = \mathbf{x}, \mathbf{U}(t) = \mathbf{u}] \quad (6) \\ &= \Big[(GG^T)(t) + [H_1(t)\mathbf{x}](\Lambda_1\sigma_1)(t)[H_1(t)\mathbf{x}]^T \\ &+ [H_2(t)\mathbf{u}](\Lambda_2\sigma_2)(t)[H_2(t)\mathbf{u}]^T + (H_3\Lambda_3\sigma_3H_3^T)(t)\Big]\, dt. \end{aligned}$$

The conditional infinitesimal moments (5) and (6) are fundamental for modeling applications.

2.1. Quadratic Performance Index

The performance index is an essential component in modeling a physical reality since it is used to determine the optimal control policy. The quadratic performance index is the quadratic, cost-to-go form:

$$\begin{aligned} V(\mathbf{X}, \mathbf{U}, t) &= \frac{1}{2}\mathbf{X}^T(t_f)S(t_f)\mathbf{X}(t_f) \quad (7) \\ &+ \frac{1}{2}\int_t^{t_f} \left[\mathbf{X}^T Q \mathbf{X} + \mathbf{U}^T R \mathbf{U}\right](\tau)d\tau, \end{aligned}$$

where the time horizon is $t_f \geq t$. The cost matrices Q and S are symmetric, positive semi-definite $m \times m$ matrices and R is a symmetric, positive definite $n \times n$ matrix. Equations (1) and (7) comprise the LQGP problem. A nontrivial terminal cost matrix S is essential for a nontrivial solution for this problem. The quadratic performance index (7) was selected to extend the results of the LQG problem.

3. Dynamic Programming Formulation

The optimal, expected performance, $v(\mathbf{x}, t)$, is defined as

$$v(\mathbf{x}, t) \equiv \underset{\mathbf{u}(t,t_f]}{\text{Min}}\, [\underset{\mathbf{P},\mathbf{W}(t,t_f]}{\text{Mean}}[V(\mathbf{X}, \mathbf{U}, t) \mid \mathbf{X}(t) = \mathbf{x}, \mathbf{U}(t) = \mathbf{u}]]. \quad (8)$$

Applying the principle of optimality to the optimal, expected performance index, (8,7), and the chain rule for Markov stochastic processes in continuous time yields the partial differential equation of stochastic dynamic programming:

$$\begin{aligned} 0 &= \frac{\partial v}{\partial t}(\mathbf{x}, t) + \underset{\mathbf{u}}{\text{Min}}\Bigg[\left(A(t)\mathbf{x} + B(t)\mathbf{u} + \mathbf{C}(t)\right)^T \nabla_x v(\mathbf{x}, t) \\ &+ \frac{1}{2}\left(GG^T\right)(t) : \nabla_x \nabla_x^T v(\mathbf{x}, t) + \frac{1}{2}\mathbf{x}^T Q(t)\mathbf{x} + \frac{1}{2}\mathbf{u}^T R(t)\mathbf{u} \\ &+ \Sigma_i \lambda_{1,i} \int_{\mathcal{Z}_{1i}} \left[v(\mathbf{x} + [H_1\mathbf{x}]_i z_i, t) - v(\mathbf{x}, t)\right] \phi_{1,i}(z_i, t)dz_i \quad (9) \\ &+ \Sigma_i \lambda_{2,i} \int_{\mathcal{Z}_{2i}} \left[v(\mathbf{x} + [H_2\mathbf{u}]_i z_i, t) - v(\mathbf{x}, t)\right] \phi_{2,i}(z_i, t)dz_i \\ &+ \Sigma_i \lambda_{3,i} \int_{\mathcal{Z}_{3i}} \left[v(\mathbf{x} + [H_3]_i z_i, t) - v(\mathbf{x}, t)\right] \phi_{3,i}(z_i, t)dz_i\Bigg], \end{aligned}$$

where $A : B = \sum_i \sum_j A_{i,j}B_{j,i} = \text{Trace}[AB^T]$. The last three terms of (9) represent the total contribution of the jump in the state from the Poisson processes. The backward partial differential equation (PDE) (9) is known as the Hamilton-Jacobi-Bellman equation and is subject to the final condition, $v(\mathbf{x}, t_f) = \frac{1}{2}\mathbf{x}^T S(t_f)\mathbf{x}$.

3.1. Formal LQGP Solution

To solve (9), assume a modification solution of the form for a LQG problem (for the usual LQG, see Bryson and Ho [5] or Lewis [11]):

$$\begin{aligned} v(\mathbf{x}, t) &= \frac{1}{2}\mathbf{x}^T S(t)\mathbf{x} + \mathbf{D}^T(t)\mathbf{x} + E(t) \\ &+ \frac{1}{2}\int_t^{t_f} \left(GG^T\right)(\tau) : S(\tau)d\tau. \quad (10) \end{aligned}$$

The modification of the assumed solution form takes into the account that the Poisson mark distribution is not centered, i.e., non-zero mean amplitudes, so a linear term with time dependent coefficient $\mathbf{D}^T(t)$ and a state-control independent term $E(t)$ have been included. The final condition of (9) is satisfied, provided that $\mathbf{D}(t_f) = \mathbf{0}$ and $E(t_f) = 0$, since symmetric, quadratic form coefficient $S(t)$ has been chosen as a backward extension of the final value $S(t_f)$ for $t < t_f$.

The regular, unconstrained optimal control, $\mathbf{u}^* = \mathbf{u}_{\text{reg}}$, is given by:

$$\mathbf{u}_{\text{reg}}(t) = -\widehat{R}^{-1}(t)\widehat{B}^T(t)\left[S(t)\mathbf{x} + \mathbf{D}(t)\right]. \tag{11}$$

Assuming regular control, the coefficients for the optimal expected performance (10) are given by,

$$\begin{aligned} 0_{m\times m} &= \dot{S}(t) + \left[A^T S + SA + Q\right](t) \\ &+ \widetilde{\Gamma}_1(t) - \left[S\widehat{B}\widehat{R}^{-1}\widehat{B}^T S\right](t), \end{aligned} \tag{12}$$

$$\begin{aligned} \mathbf{0}_{m\times 1} &= \dot{\mathbf{D}}(t) + \left[\left(A + (\Lambda_1\overline{\mathbf{Z}}_1)^T H_1^T\right)^T \mathbf{D}\right](t) \\ &+ \left[S\left(\mathbf{C} + H_3\Lambda_3\overline{\mathbf{Z}}_3\right) - S\widehat{B}\widehat{R}^{-1}\widehat{B}^T\mathbf{D}\right](t), \end{aligned} \tag{13}$$

and

$$\begin{aligned} 0 &= \dot{E}(t) + \left[\left(\mathbf{C} + H_3\Lambda_3\overline{\mathbf{Z}}_3\right)^T \mathbf{D}\right](t) \\ &+ \frac{1}{2}\left[\left(H_3^T S H_3\right) : \Lambda_3\overline{ZZ}_3 - \mathbf{D}^T\widehat{B}\widehat{R}^{-1}\widehat{B}^T\mathbf{D}\right](t) \end{aligned} \tag{14}$$

where

$$\begin{aligned} \Gamma_1(t) &\equiv \left[\left([H_1^T]_i S[H_1]_j : \Lambda_1\overline{ZZ}_1\right)(t)\right]_{m\times m} \\ &+ 2\left[\left(\Lambda_1\overline{\mathbf{Z}}_1\right)^T H_1^T S\right](t), \end{aligned} \tag{15}$$

$$\Gamma_2(t) \equiv \left[\left([H_2^T]_i S[H_2]_j : \Lambda_2\overline{ZZ}_2\right)(t)\right]_{n\times n}, \tag{16}$$

$$\overline{ZZ}_\ell(t) \equiv \overline{\sigma}_\ell(t) + \left(\overline{\mathbf{Z}}_\ell\overline{\mathbf{Z}}_\ell^T\right)(t) = \left[\sigma_{\ell,i}\delta_{i,j} + \overline{Z}_{\ell,i}\overline{Z}_{\ell,j}\right]_{q_\ell\times q_\ell}$$

for $\ell = 1$ to 3 with $\widehat{R}(t) \equiv R(t) + \widetilde{\Gamma}_2(t)$, $\widehat{B}(t) \equiv B(t) + ((\Lambda_2\overline{\mathbf{Z}}_2)^T H_2^T)(t)$, and $\widetilde{\Gamma}_\ell = (\Gamma_\ell + \Gamma_\ell^T)$. Since the matrix R is positive definite, R^{-1} exists and then so does $\widehat{R}^{-1}$. Note (12) appears to have Riccati-like quadratic form, but in general is highly nonlinear through the S dependence of $\widehat{R}$. If $H_\ell = [H_{\ell,i,j,k}]_{m\times q_\ell\times m_\ell}$, then $H_\ell^T = [H_{\ell,j,i,k}]_{q_\ell\times m\times m_\ell}$. Note that the quadratic coefficient of the solution form, $S(t)$, appears in the above coefficient definitions (15,16). This added complexity of the space-time Poisson distribution means added realism for the model.

The optimal control in (11) is a feedback control. However, this feedback control is not purely linear in the state $\mathbf{x}$ as it would be without the nonzero mean Poisson amplitude, but is an affine function of the state, i.e., a linear function plus state independent part. If there are control constraints, then the assumed form of the solution becomes more complicated than the general quadratic LQGP form in (10). For example, in the case of component-wise or hypercube constraints, $U_{\min,i}(t) \leq u_i(t) \leq U_{\max,i}(t)$, the optimal control takes on a piecewise form: $U_i^*(t) = \min[U_{\max,i}(t), \max[U_{\min,i}(t), u_{\text{reg},i}(t)]]$.

Due to uni-directional coupling of these matrix differential equations, it is assumed that the nonlinear matrix differential equation (12) for $S(t)$ is solved first and the result for $S(t)$ is substituted into equation (13) for $\mathbf{D}(t)$, which is then solved, and then both results for $S(t)$ and $\mathbf{D}(t)$ are substituted into equation (14) for the state-control independent term $E(t)$. Since $S(t)$ is a symmetric matrix by being defined with a quadratic form, only a triangle part of $S(t)$ need be solved, or $n\cdot(n+1)/2$ component equations. Thus, for the whole coefficient set $\{S(t), \mathbf{D}(t), E(t)\}$, only $n\cdot(n+1)/2 + n + 1$ component equations need to be solve, so that for large n the count is $\mathcal{O}(n^2/2)$, asymptotically, which is the same order of effort in getting the triangular part of $S(t)$.

4. Multistage Manufacturing System Example

Consider a simple multistage manufacturing system (MMS) that produces a single consumable good. In each of the k stages of the manufacturing process a component or sub-assembly of the final product is added. Assume that the production flow follows a linear sequence, i.e., a part that has completed i stages is the input material for stage $i+1$. The initial, loading stage is where all of the raw materials for all stages are input into the system. The final, unloading stage is the mechanism by which the final product is delivered to the consumer. For simplicity, the loading and unloading stages are not considered as part of the MMS. This model is similar to a flexible manufacturing system (FMS), but its perspective is global instead of local. Each stage can be viewed as an FMS. Kimemia and Gershwin [10] describe the similarities and differences between FMS and MMS, while presenting an algorithm for FMS control. A survey of many types of real flexible manufacturing systems is given by Dupont-Gatelmand [6].

At time t in the planning horizon while in stage i, there are $n_i(t)$ operational workstations. For each stage i, all workstations are assumed to have identical properties and produce at the same rate $c_i(t)$ with a capacity of producing M_i parts per unit time. The production rate $c_i(t)$ is the utilization, i.e., the fraction of time busy, of the workstations at stage i and is bounded by $0 \leq c_i(t) \leq c_{\max,i}(t)$, where $c_{\max,i}(t)$, is the minimum of 1 and $c_{i-1}(t) * M_{i-1} * n_{i-1}(t)/(M_i * n_i(t))$, due to the physical and production limitations, respectively. Each workstation is subject to failure and can be repaired. The mean time between failures and the repair duration are exponentially distributed. For similar models with variations see Akella and Kumar [1] for a treatment of optimal inventory levels, as well as Boukas and co-workers [3, 4] for a treatment that includes preventive maintenance and machine age structure.

The total number of workstations at stage i is N_i. Hence, $n_i(t)$ can be viewed as a birth and death process confined to the interval $0 \leq n_i(t) \leq N_i$. The defining equation for the number of operational workstations is given by

$$dn_i(t) = dP_i^R(t) - dP_i^F(t), \tag{17}$$

where $dP_i^R(t)$ and $dP_i^F(t)$ are Poisson processes used to model the repair (birth) processes and the failure (death) processes, respectively.

The state of the manufacturing system is described by the number of operational workstations, $n_i(t)$, and by the surplus aggregate level, $a_i(t)$. The surplus aggregate level represents the surplus (if positive) or shortfall (if negative) of the production of pieces that have successfully completed i stages of the manufacturing process. The state equation for the surplus aggregate level for stage $i = 1$ to k is given by

$$da_i(t) = \left[M_i c_i(t) n_i(t) + u_i(t) - d_i(t)\right] dt + g_i(t) dW_i(t). \tag{18}$$

The change in the surplus aggregate level, $da_i(t)$, is determined by the number of pieces that have successfully completed i stages of the manufacturing process ($M_i n_i(t) c_i(t) dt$), that are not defective, and are not consumed by stage $i+1$ ($d_i(t)dt$), and by the status of the workstations. The term $u_i(t)dt$ is used to adjust the production rate, $c_i(t)$, where the control $u_i(t)$ is expressed as the number of pieces per unit time. The last term, $g_i(t)dW_i(t)$, is used to model the random fluctuations in the number of defective pieces.

The regular control production rate is defined as: $c_{\text{reg},i}(t) = VSPR_i(t) + u_{\text{reg},i}(t)/(M_i n_i(t))$, where the virtual static production rate is defined as: $VSPR_i(t) = d_i(t)/(M_i n_i(t))$. The control is bounded so that the production rate is valid which leads to similiar expression for the constrained control production rate.

The cost functional to be minimized is a quadratic cost-to-go form given by (7) with $Q(t) \equiv 0$ where the state or plant vector is given by: $\mathbf{X}(t) = \left[\mathbf{a}(t) \quad \mathbf{n}(t)\right]^T$. The salvage cost matrix S is a symmetric, positive semi-definite $2k \times 2k$ matrix. The salvage cost is based on the idea of *Just In Time* or *stockless production* (see Hall [8]). This means that a penalty is imposed if we have a surplus or shortfall of production at the end of the planning horizon in any stage of the manufacturing system. Further motivation for the salvage term is given by Bielecki and Kumar [2], who show that, for an unreliable manufacturing system, the optimal policy is a zero inventory policy.

For numerical concreteness, consider a MMS with $k = 3$ stages with a planning horizon of 40 hours. Let the initial surplus aggregate level for all stages be zero. Let the demand be $d_i(t) = 285$ pieces per hours for $i = 1$ to 3. The Gaussian random fluctuations of production will be assumed absent, i.e., $g_i(t) = 0$ for $i = 1$ to 3. Let the total number of workstations, $N_i(t)$, for each stage be 3, 5, and 4, respectively. The salvage cost and the instantaneous quadratic cost coefficient matrices are given by:

$$S(t_f) = \begin{bmatrix} S_1(t_f) & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix}, S_1(t_f) = \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 1.9 & 0 \\ 0 & 0 & 2.6 \end{bmatrix}$$

and

$$R(t) = \begin{bmatrix} 1.3E4 & 0 & 0 \\ 0 & 0.8E4 & 0 \\ 0 & 0 & 0.6E4 \end{bmatrix}.$$

The individual characteristics for an individual given workstation is summarized in the table below.

Stage i	Production Capacity, M_i (pieces/hour)	Mean Time between Failure $1/\lambda_i^F$ (hours)	Mean Time to Repair $1/\lambda_i^R$ (hours)
1	117	25.0	2.5
2	71	21.5	4.0
3	88	30.0	1.5

In the case of unconstrained control, the regular control is given by:

$$\mathbf{u}_{\text{reg}}(t) = -R^{-1}(t)B^T(t)\left[S(t)\mathbf{x} + \mathbf{D}(t)\right]. \tag{19}$$

The quadratic matrix coefficient of the optimal value becomes a genuine Riccati matrix equation,

$$\begin{aligned} 0_{2k\times2k} &= \dot{S}(t) + A^T(t)S(t) + S(t)A(t) \\ &- S(t)B(t)R^{-1}(t)B^T(t)S(t), \end{aligned} \tag{20}$$

the equations for the linear and state-independent coefficients become

$$\begin{aligned} \mathbf{0}_{2k\times1} &= \dot{\mathbf{D}}(t) + S(t)\left[\mathbf{C}(t) + H_3\Lambda_3(t)\overline{\mathbf{Z}}_3(t)\right] \\ &+ A^T(t)\mathbf{D}(t) - S(t)B(t)R^{-1}(t)B^T(t)\mathbf{D}(t) \end{aligned} \tag{21}$$

and

$$\begin{aligned} 0 &= \dot{E}(t) + \left[\left(\mathbf{C} + H_3\Lambda_3\overline{\mathbf{Z}}_3\right)^T \mathbf{D}\right](t) \\ &+ \frac{1}{2}\left[\left(H_3^T S H_3\right) : \Lambda_3\overline{ZZ}_3\right](t) \\ &- \frac{1}{2}\left[\mathbf{D}^T B R^{-1} B^T \mathbf{D}\right](t), \end{aligned} \tag{22}$$

where

$$A(t) = \begin{bmatrix} 0_{3\times3} & \text{diag}[\mathbf{M}]\text{diag}[\mathbf{c}(t)] \\ 0_{3\times3} & 0_{3\times3} \end{bmatrix}, \; B(t) = \begin{bmatrix} I_{3\times3} \\ 0_{3\times3} \end{bmatrix},$$

$$\mathbf{C}(t) = \begin{bmatrix} -\mathbf{d}(t) \\ \mathbf{0}_{3\times1} \end{bmatrix}, \; H_3(t) = \begin{bmatrix} 0_{3\times3} & 0_{3\times3} \\ I_{3\times3} & -I_{3\times3} \end{bmatrix},$$

and

$$d\mathbf{P}_1(t) = \mathbf{0}, \; d\mathbf{P}_2(t) = \mathbf{0}, \; d\mathbf{P}_3(t) = \begin{bmatrix} d\mathbf{P}^R(t) \\ d\mathbf{P}^F(t) \end{bmatrix},$$

with $\text{diag}[\mathbf{c}(t)] = [\Sigma_j \delta_{i,j} c_j]_{k\times1}$ being the diagonalization of the vector whose components are given by $c_i(t)$ for $i = 1$ to k.

The figures are for the second stage in the manufacturing process with [3,4,*] active workstations. In Figure 1, regular control and constrained control production rates, $c_2(t)$, exhibit the anticipation of workstation repair and failure, whereas the maximal, virtual static production rate does not. The constrained control production rate saturates at 100% capacity, while the unconstrained, regular control production rate exceeds 100% capacity. In Figure 2, the projected surplus aggregate level, $a_2(t)$, becomes negative when the constrained control production rate, $c_2(t)$, saturates at 100% of capacity.

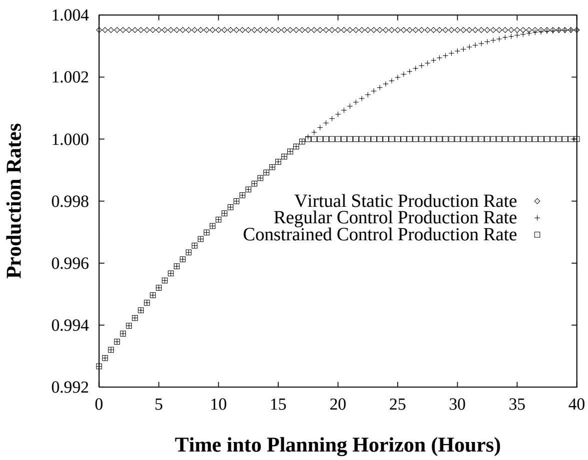

Figure 1: Virtual static, regular control and constrained control production rates, $c_2(t)$, for stage 2 with [3,4,*] active workstations.

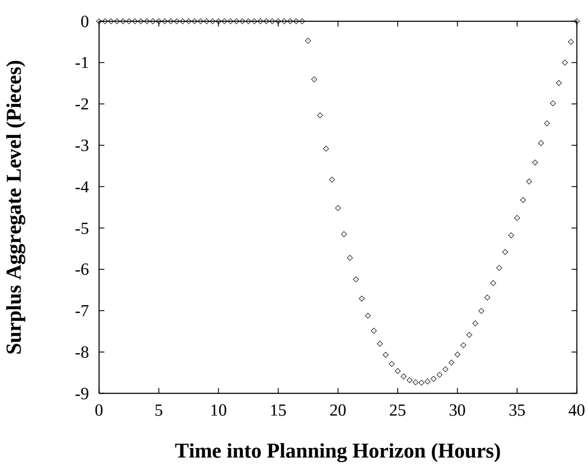

Figure 2: The surplus aggregate level, $a_2(t)$, for stage 2 with [3,4,*] active workstations.

5. Conclusions

These results are of great interest because the LQGP problem is a theoretical model and a canonical computational model having several exact analytical results, except for solving the LQGP nonlinear equation (12), and the auxiliary equations (13,14). Also, the problem will be useful for severe testing of physical devices in computer experiments, investigating large fluctuations in financial markets, as well as many other applications, because the Poisson random component is useful for modeling large random fluctuations, whereas Gaussian noise is useful for relatively milder background fluctuations. Also, the stochastic differential equation formulation used here makes the construction of models analogous to that for other dynamic systems. The multistage manufacturing system application is a good test of our methods.

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The first author (JJW) dedicates this work to the memory of his parents Marie L. and Robert J. Westman.

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