The NLQGP Problem: Application to a Multistage Manufacturing System *

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Abstract

The Nonlinear Quadratic Gaussian Poisson (NLQGP) problem denotes an optimal control problem with nonlinear dynamics and quadratic costs with both Gaussian and Poisson noise disturbances. The NLQGP problem provides a comprehensive model for many applications since the noises considered are quite robust and add extra realism to physical models. The problem is examined and is illustrated with an application to a multistage manufacturing system (MMS) in an uncertain environment.

1 Introduction

The nonlinear dynamics, quadratic performance, Gaussian noise and Poisson noise or NLQGP problem, has its dynamics governed by the stochastic differential equation (SDE)

$$\begin{aligned} d\mathbf{X}(t) &= [\mathbf{F_0}(\mathbf{X}(t),t) + F_1(\mathbf{X}(t),t)\mathbf{U}(t)]dt + G(t)d\mathbf{W}(t) \\ &+ H_0(\mathbf{X}(t),t)d\mathbf{P}_0(t) + [H_1(\mathbf{X}(t),t)\mathbf{U}(t)]d\mathbf{P}_1(t) \end{aligned} \quad (1)$$

for general Markov processes in continuous time, with $m\times 1$ state vector $\mathbf{X}(\mathrm{t})$, $n\times 1$ control vector $\mathbf{U}(\mathrm{t})$, $r\times 1$ Gaussian noise vector $\mathrm{d}\mathbf{W}(\mathrm{t})$, and $q_\ell\times 1$ space-time Poisson noise vectors $d\mathbf{P}_\ell(t)$, for $\ell = 0$ or 1. The dynamics, with respect to the state, evolve according to the nonlinear vector valued function $\mathbf{F_0}(\mathbf{X}(t),t)$ which is $m\times 1$. The dimensions of the respective coefficient matrices are: $F_1(t)$ is $m\times n$, $G(t)$ is $m\times r$, $H_0(\mathbf{X}(t),t)$ is $m\times q_0$ and $H_1(\mathbf{X}(t),t)$ is dimensioned so that the nonstandard linear algebra form, $[H_1(\mathbf{X}(t),t)\mathbf{U}(\mathbf{t})] = [\sum_{k=1}^{r} H_{1,ijk}(\mathbf{X}(t),t)U_k(t)]$, is $m\times q_1$. Note that the state dependence is assumed to be in general nonlinear, such as in the coefficient of the $d\mathbf{P}_0$ term, while the coefficient of the $d\mathbf{P}_1$ is bilinear in the control vector $\mathbf{U}$ and a nonlinear function of the state vector $\mathbf{X}$. It is assumed that the SDE (1) is interpreted in the sense of Itô.

The NLQGP problem advances the model of the LQGP problem [19] which is an extension of the LQG analysis (see Bryson and Ho [5] or Lewis [15]). In contrast to the LQGP problem, the NLQGP problem does not exhibit a closed form solution and numerical methods are necessary in order to solve the partial

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differential equation of stochastic dynamic programming. Because of this, the computational requirements for the NLQGP problem, a search over the state space, is much greater than that for LQGP problem, following the evolution of the coefficient space.

The NLQGP problem is similar to the *LQG in control only* model used by Hanson [11, 12] and Naimipour and Hanson [16]. The main difference is that the NLQGP problem has a more robust representation of the Poisson noises. In [11, 16, 12], the Poisson noise used is a simpler form such that the Poisson coefficient depends only on the state, though nonlinearly; whereas the NLQGP problem uses a compound Poisson noise that is bilinear in the control and a nonlinear function of the state. This bilinear term increases not only the realism that can be included in a model, but greatly increases the computational requirements needed to solve the partial differential equation of stochastic dynamic programming. The Poisson terms of (1) introduce a discrete jump in the state, superimposed on the continuous, yet nonsmooth Gaussian noise contribution. The more general and current name for combinations of discrete (e.g., Markov chains or Poisson noise) and continuous systems is *hybrid system*.

The paper is arranged as follows. In Section 2, the the first and second moments of the SDE (1) are derived. In Section 3, the quadratic performance index and the stochastic dynamic programming formulation are described. In Section 4, the method for the numerical solution for the Hamilton-Jacobi-Bellman equation is examined. In Section 5, a computational example for a multistage manufacturing system (MMS) is explored.

2 Infinitesimal Conditional Moments

The Gaussian white noise term, $d\mathbf{W}(t)$, consists of r independent, standard Wiener processes $dW_i(t)$, for $i = 1$ to r. It is assumed that the Gaussian components have zero infinitesimal mean and diagonal covariance,

$$\mathrm{Mean}[d\mathbf{W}(t)] = \mathbf{0}_{\mathbf{r}\times\mathbf{1}} \quad \text{and} \quad \mathrm{Covar}[d\mathbf{W}(t), d\mathbf{W}^T(t)] = I_r dt. \tag{2}$$

For cross-correlated noise, see Stengel [18]. We have left the Gaussian noise coefficient independent of both state and control vectors for simplicity, but $G(t)$ could well be generalized like those of the Poisson noise coefficients, $H_0(\mathbf{X}(t), t)$ and $[H_1(\mathbf{X}(t), t)\mathbf{U}(t)]$.

The space-time Poisson white noise terms, $d\mathbf{P}_\ell(t) = [dP_{\ell,i}(t)]_{q_\ell \times 1}$, consist of q_ℓ independent differentials of space-time Poisson processes each. They are related to Poisson random measure, $\mathcal{P}_{\ell,i}(dz_i, dt)$, formulation (see Gihman and Skorohod [9]):

$$dP_{\ell,i}(t) = \int_{\mathcal{Z}_{\ell,i}} z_i \mathcal{P}_{\ell,i}(dz_i, dt), \tag{3}$$

where z_i is the Poisson amplitude random variable or mark of the $dP_{\ell,i}(t)$ processes, for $\ell = 0$ or 1 while $i = 1$ to q_ℓ. These have mean

$$\mathrm{Mean}[d\mathbf{P}_\ell(t)] = \Lambda_\ell(t)dt \int_{\mathcal{Z}_\ell} \mathbf{z}\phi_\ell(\mathbf{z}, t)d\mathbf{z} = \Lambda_\ell(t)dt \left[\int_{\mathcal{Z}_{\ell,i}} z_i \phi_{\ell,i} dz_i\right]_{q_\ell \times 1} \equiv \Lambda_\ell \overline{\mathbf{Z}}_\ell dt, \tag{4}$$

where $\Lambda_\ell(t)$ is the diagonal representation of the Poisson rates $\lambda_{\ell,i}(t)$, $\overline{\mathbf{Z}}_\ell(t)$ is the vector mean of the amplitude mark and $\phi_{\ell,i}(z_i, t)$ is the density of the $\{\ell, i\}$th amplitude mark component, and covariance assuming component-wise independence

$$\begin{aligned} \mathrm{Covar}[d\mathbf{P}_\ell(t)] &= \Lambda_\ell(t)dt \int_{\mathcal{Z}_\ell} (\mathbf{z} - \overline{\mathbf{Z}}_\ell)(\mathbf{z} - \overline{\mathbf{Z}}_\ell)^T \phi_\ell(\mathbf{z}, t)d\mathbf{z} \\ &= \Lambda_\ell(t)dt \left[\int_{\mathcal{Z}_{\ell,i}} (z_i - \overline{Z}_{\ell,i})^2 \phi_{\ell,i} dz_i\right]_{q_\ell \times 1} \\ &\equiv \Lambda_\ell(t)\sigma_\ell(t)dt = [\lambda_{\ell,i}\sigma_{\ell,ij}\delta_{ij}dt]_{q_\ell \times q_\ell}, \end{aligned} \tag{5}$$

with $\sigma_\ell(t)$ denoting the diagonalized variance of the amplitude mark distribution for $\mathbf{P}_\ell(t)$. If the mark vector density does not exist even in terms of generalized functions, than $\phi_\ell(\mathbf{z}, t)d\mathbf{z}$ could be easily replaced by the corresponding distribution $\Phi_\ell(d\mathbf{z},t)$. Also, note that the mark distributions have not been assumed to be zero mean distributions, i.e., $\overline{\mathbf{Z}}_\ell \neq 0$, so that there will be some added complexity. It is further assumed that all of the individual component terms of the Gaussian and Poisson noises are independent, i.e.,

$$\text{Covar}[d\mathbf{W}(t), d\mathbf{P}_\ell^T(t)] = 0_{r\times q_\ell}. \tag{6}$$

The jth jump of the $\{\ell,i\}$th space-time Poisson process at time $t_{\ell,i,j}$ with amplitude $z_{\ell,i,j}$ causes the following jump from $t^-_{\ell,i,j}$ to $t^+_{\ell,i,j}$ in the state:

$$[\mathbf{X}](t_{\ell,i,j}) = \begin{cases} [[H_0(\mathbf{X}(t^-_{\ell,i,j}), t_{\ell,i,j})]_{i,k} z_{\ell,i,j}]_{m\times 1}, & \ell = 0 \\ [[H_1(\mathbf{X}(t^-_{\ell,i,j}), t_{\ell,i,j})\mathbf{U}(t^-_{\ell,i,j})]_{i,k} z_{\ell,i,j}]_{m\times 1}, & \ell = 1 \end{cases}. \tag{7}$$

From the above statistical properties of the stochastic processes, $d\mathbf{W}$ and $d\mathbf{P}_\ell$, it follows that the conditional infinitesimal expectation of the state is

$$\begin{aligned} &\text{Mean}[d\mathbf{X}(t) \mid \mathbf{X}(t) = \mathbf{x}, \mathbf{U}(t) = \mathbf{u}] \\ &\quad = [F_0(\mathbf{x},t) + F_1(\mathbf{x},t)\mathbf{u} + H_0(\mathbf{x},t)\Lambda_0\overline{\mathbf{Z}}_0(t) + [H_1(\mathbf{x},t)\mathbf{u}]\Lambda_1\overline{\mathbf{Z}}_1(t)]dt \end{aligned} \tag{8}$$

and the conditional infinitesimal covariance,

$$\begin{aligned} &\text{Covar}[d\mathbf{X}(t) \mid \mathbf{X}(t) = \mathbf{x}, \mathbf{U}(t) = \mathbf{u}] \\ &= \left[(GG^T)(t) + H_0(\mathbf{x},t)(\Lambda_0\sigma_0)(t)H_0^T(\mathbf{x},t) + [H_1(\mathbf{x},t)\mathbf{u}](\Lambda_1\sigma_1)(t)[H_1(\mathbf{x},t)\mathbf{u}]^T\right] dt. \end{aligned} \tag{9}$$

The conditional infinitesimal moments (8) and (9) are fundamental for modeling applications. Note that the conditional infinitesimal mean (8) is determined by the nonlinear dynamics and the contributions from the Poisson noise processes, whereas the conditional infinitesimal covariance (9) is determined exclusively from the Gaussian and Poisson noise processes.

3 Quadratic Performance Index and Stochastic Dynamic Programming Formulation

A performance index is a measure used to decide how the system should behave in an optimal way, e.g., a tracking problem may use the consumption of fuel or time as a performance index. The performance index is an essential component in modeling a physical reality since it is used to determine the optimal control policy. The quadratic performance index considered here is the quadratic, cost-to-go form:

$$\begin{aligned} V(\mathbf{X},\mathbf{U},t) &= \frac{1}{2}\mathbf{X}^T(t_f)S(t_f)\mathbf{X}(t_f) \\ &+ \int_t^{t_f} \left[C_0(\mathbf{X},\tau) + \mathbf{C}_1^T(\mathbf{X},\tau)\mathbf{U}(\tau) + \frac{1}{2}\mathbf{U}^T(\tau)C_2(\mathbf{X},\tau)\mathbf{U}(\tau)\right] d\tau, \end{aligned} \tag{10}$$

where the time horizon is $t_f \geq t$. The salvage cost matrix S is a symmetric, positive semi-definite $m \times m$ matrix and C_2 is a symmetric, positive definite $n\times n$ matrix where the cost coefficients C_1 is a $n\times 1$ vector and C_0 is a scalar. The quadratic performance index (10) was selected to be the most general form. Equations (1) and (10) comprise the NLQGP problem. The final condition,

$$V(\mathbf{X}(t_f), \mathbf{U}(t_f), t_f) = \frac{1}{2}\mathbf{X}^T(t_f)S(t_f)\mathbf{X}(t_f) \tag{11}$$

yields the salvage cost. The performance index (10) is a combination of the salvage cost and the instantaneous cost, the integral term of (10).

The optimal, expected performance, $v(\mathbf{x},t)$, is defined as

$$v(\mathbf{x},t) \equiv \underset{\mathbf{u}(t,t_f]}{\text{Min}} \left[\underset{\mathbf{P},\mathbf{W}(t,t_f]}{\text{Mean}} [V(\mathbf{X},\mathbf{U},t) \mid \mathbf{X}(t) = \mathbf{x}, \mathbf{U}(t) = \mathbf{u}]\right]. \tag{12}$$

The restrictions on the state and control are that they belong to the admissible classes for the state and control, respectively.

Applying the principle of optimality to the optimal, expected performance index, (12,10), and the chain rule for Markov stochastic processes in continuous time [9, 12],

$$\begin{aligned}
dv(\mathbf{X}(t),t) &= v(\mathbf{X}(t+dt),t+dt) - v(\mathbf{X}(t),t) \\
&= \left[\frac{\partial v}{\partial t}(\mathbf{X}(t),t) + (\mathbf{F}_0(\mathbf{X}(t),t) + F_1(\mathbf{X}(t),t)\mathbf{U}(t))^T \nabla_x v(\mathbf{X}(t),t)\right. \\
&+ \left.\frac{1}{2}(GG^T)(t) : \nabla_x \nabla_x^T v(\mathbf{X}(t),t)\right] dt + (Gd\mathbf{W})^T(t) \nabla_x v(\mathbf{X}(t),t) \\
&+ \int_{\mathcal{Z}_0} \left[v(\mathbf{X}(t) + H_0(\mathbf{X}(t),t)\mathbf{z},t) - v(\mathbf{X}(t),t)\right] \mathcal{P}_0(d\mathbf{z},dt) \\
&+ \int_{\mathcal{Z}_1} \left[v(\mathbf{X}(t) + [H_1(\mathbf{X}(t),t)\mathbf{U}(t)]\mathbf{z},t) - v(\mathbf{X}(t),t)\right] \mathcal{P}_1(d\mathbf{z},dt)
\end{aligned} \tag{13}$$

yields the partial differential equation of stochastic dynamic programming, where

$$A : B = \sum_i \sum_j A_{i,j} B_{j,i} = \mathrm{Trace}[AB^T]. \tag{14}$$

The last two terms of (13) represents the total contributions of the jumps in the state from the Poisson processes. Hence, the partial differential equation of stochastic dynamic programming is:

$$\begin{aligned}
0 &= \frac{\partial v}{\partial t}(\mathbf{x},t) + \min_{\mathbf{u}} \left[(\mathbf{F}_0(\mathbf{x},t) + F_1(\mathbf{x},t)\mathbf{u})^T \nabla_x v(\mathbf{x},t)\right. \\
&+ \frac{1}{2}\left(GG^T\right)(t) : \nabla_x \nabla_x^T v(\mathbf{x},t) + C_0(\mathbf{x},t) + \mathbf{C}_1{}^T(\mathbf{x},t)\mathbf{u} + \frac{1}{2}\mathbf{u}^T C_2(\mathbf{x},t)\mathbf{u} \\
&+ \sum_{j=1}^{q_0} \lambda_{0,j} \int_{\mathcal{Z}_{0j}} \left[v(\mathbf{x} + [H_{0,ij}(\mathbf{x},t)]_{m\times 1} z_j, t) - v(\mathbf{x},t)\right] \phi_{0,j}(z_j,t) dz_j \\
&+ \left.\sum_{j=1}^{q_1} \lambda_{1,j} \int_{\mathcal{Z}_{1j}} \left[v(\mathbf{x} + [[H_1(\mathbf{x},t)\mathbf{u}]_{ij}]_{m\times 1} z_j, t) - v(\mathbf{x},t)\right] \phi_{1,j}(z_j,t) dz_j\right].
\end{aligned} \tag{15}$$

The backward partial differential equation (PDE) (15) is known as the Hamilton-Jacobi-Bellman (HJB) equation and is subject to the final condition:

$$v(\mathbf{x},t_f) = \frac{1}{2}\mathbf{x}^T S(t_f)\mathbf{x}. \tag{16}$$

The argument of the minimum function of (15) is called the Hamiltonian of the system. The HJB equation (15) can be rewritten as follows:

$$\begin{aligned}
0 &= \frac{\partial v}{\partial t}(\mathbf{x},t) + \mathbf{F}_0{}^T(\mathbf{x},t)\nabla_x v(\mathbf{x},t) \\
&+ \frac{1}{2}(GG^T)(t) : \nabla_x \nabla_x^T v(\mathbf{x},t) + C_0(\mathbf{x},t) + \mathcal{S}^*(\mathbf{x},t) \\
&+ \sum_{j=1}^{q_0} \lambda_{0,j} \int_{\mathcal{Z}_{0j}} \left[v(\mathbf{x} + [H_{0,ij}(\mathbf{x},t) z_j]_{m\times 1}, t) - v(\mathbf{x},t)\right] \phi_{0,j}(z_j,t) dz_j,
\end{aligned} \tag{17}$$

where the control switching term, $\mathcal{S}^*(\mathbf{x},t)$, is given by,

$$\begin{aligned}
\mathcal{S}^*(\mathbf{x},t) &= \min_{\mathbf{u}} \left[(F_1(\mathbf{x},t)\mathbf{u})^T \nabla_x v(\mathbf{x},t) + \mathbf{C}_1{}^T(\mathbf{x},t)\mathbf{u} + \frac{1}{2}\mathbf{u}^T C_2(\mathbf{x},t)\mathbf{u}\right. \\
&+ \left.\sum_{j=1}^{q_1} \lambda_{1,j} \int_{\mathcal{Z}_{1j}} \left[v(\mathbf{x} + [[H_1(\mathbf{x},t)\mathbf{u}]_{ij} z_j]_{m\times 1}, t) - v(\mathbf{x},t)\right] \phi_{1,j}(z_j,t) dz_j\right].
\end{aligned} \tag{18}$$

The argument of minimum (18) is known as the optimal regular (i.e. unconstrained) control, $\mathbf{u}^{\mathrm{reg}}(t)$. However, due to the complexity of the control dependence in the integral of (18), we can no longer explicitly solve for $\mathbf{u}^{\mathrm{reg}}(t)$ as in [19], but get the more complicated implicit form:

$$\mathbf{u}^{\mathrm{reg}}(t) = -C_2^{-1}\left(F_1^T\nabla_x v + C_1 + \sum_{j=1}^{q_1}\lambda_{1,j}\int_{\mathcal{Z}_{1j}}\nabla_x^T[v](\mathbf{x}+[[H_1\mathbf{u}^{\mathrm{reg}}]_{ij}]_{m\times 1}z_j,t)[H_{1,ijk}]_{m\times r}z_j\phi_{1,j}(z_j,t)dz_j\right). \quad (19)$$

In the case of component-wise or hypercube constraints,

$$U_i^{\min}(t) \le u_i^*(t) \le U_i^{\max}(t), \quad (20)$$

the optimal control, $\mathbf{u}^*(t)$, is determined by:

$$u_i^*(t) = \begin{cases} U_i^{\min}(t) & u_i^{\mathrm{reg}}(t) < U_i^{\min}(t) \\ u_i^{\mathrm{reg}}(t) & U_i^{\min}(t) \le u_i^{\mathrm{reg}}(t) \le U_i^{\max}(t) \\ U_i^{\max}(t) & U_i^{\max}(t) < u_i^{\mathrm{reg}}(t) \end{cases}. \quad (21)$$

4 Numerical Considerations for the Solution of the Hamilton-Jacobi-Bellman Equation

The system that needs to be solved is the backward PDE of stochastic dynamic programming (17) subject to the final condition (16). Additionally, the regular or unconstrained control, $\mathbf{u}^{\mathrm{reg}}(t)$, or the optimal control, $\mathbf{u}^*(t)$, if control constraints are present, is determined from the switching term (18). A summary of the discretization scheme in state and backward time, using a compact subscript notation is given below:

$$\begin{aligned}
&\mathbf{x} \to \mathbf{X_j} = [X_{ij_i}]_{m\times 1} = [X_{i0} + j_i \cdot DX_i]_{m\times 1}, \\
&\mathbf{j} = [j_i]_{m\times 1}, \text{ where } j_i = 0 \text{ to } M_i, \text{ for } i = 1 \text{ to } m; \\
&t \to T_k = t_f - k * DT \text{ for } k = 0 \text{ to } K; \\
&v(\mathbf{X_j}, T_k) \to V_{\mathbf{j}k}; \\
&\tfrac{\partial v}{\partial t}(\mathbf{X_j}, T_k) \to (V_{\mathbf{j},k+\frac{1}{2}+\frac{1}{2}} - V_{\mathbf{j},k+\frac{1}{2}-\frac{1}{2}})/(-2\cdot\tfrac{1}{2}DT); \\
&\nabla_x v(\mathbf{X_j}, T_k) \to \mathbf{DV}_{\mathbf{j}k}; \\
&\nabla_x\nabla_x^T v(\mathbf{X_j}, T_k) \to DDV_{\mathbf{j}k}; \\
&\Big[v(\mathbf{X_j} + [H_{0,ij}(\mathbf{X_j}, T_k)z_j\Big]_{m\times 1}, T_k)]_{q_0\times 1} \to \mathbf{HV0}_{\mathbf{j}k} = [HV0_{i,\mathbf{j}k}]_{q_0\times 1}; \\
&\Big[v(\mathbf{X_j} + [[H_1(\mathbf{X_j}, T_k)\mathbf{u}]_{ij}z_j]_{m\times 1}, T_k)\Big]_{q_1\times 1} \to \mathbf{HV1}_{\mathbf{j}k} = [HV1_{i,\mathbf{j}k}]_{q_1\times 1}; \\
&F_\ell(\mathbf{X_j}, T_k) \to F\{\ell\}_{\mathbf{j}k} \text{ for } \ell = 0, 1; \\
&C_\ell(\mathbf{X_j}, T_k) \to C\{\ell\}_{\mathbf{j}k} \text{ for } \ell = 0, 1, 2; \\
&\mathcal{S}^*(\mathbf{X_j}, T_k) \to \mathcal{S}_{\mathbf{j}k} = \operatorname*{Min}_{\mathbf{u}\in\mathcal{D}_\mathbf{u}}\Big[(F1_{\mathbf{j}k}\mathbf{u})^T\mathbf{DV}_{\mathbf{j}k}; + \mathbf{C1}_{\mathbf{j}k}^T\mathbf{u}(t) + \tfrac{1}{2}\mathbf{u}^T(t)C2_{\mathbf{j}k}\mathbf{u}(t) \\
&\qquad + \textstyle\sum_{i=1}^{q_1}\lambda_{1,i}\int_{\mathcal{Z}_{1i}}\Big[HV1_{i,\mathbf{j}k} - V_{\mathbf{j}k}\Big]\phi_{1,i}(z_i, T_k)dz_i\Big].
\end{aligned} \quad (22)$$

Second order central differencing is used to compute the approximated derivatives of $v(\mathbf{x},t)$ with respect to $\mathbf{x}$ at interior points, with one-sided differences of the same order for boundary points. The delay terms associated with the Poisson noises, $\mathbf{HV0}_{\mathbf{j}_k}$ and $\mathbf{HV1}_{\mathbf{j}_k}$, need to be computed using interpolation that has the same order of accuracy as that used for the derivatives.

This system can not be solved exactly, and numerical methods must be employed in order to determine the optimal, expected performance $v(\mathbf{x},t)$ and optimal control vector $\mathbf{u}^*(\mathbf{x},t)$. In the case of linear dynamics, quadratic costs and Gaussian noise (the LQG problem, see Bryson and Ho [5] or Lewis [15]) formal solutions exist that require numerical solution for a matrix Ricatti equation, additionally if there is Poisson noise (the LQGP problem, see Westman and Hanson [19]) formal solutions exist that require the numerical solution for a system on nonlinear differential equations. In order to attain a numerical solution a Crank-Nicolson, predictor–corrector method must be used since the quantities that must be determined are functionally dependent on each other. The Crank-Nicolson, extrapolated–predictor–corrector method is a modification of the method for stochastic dynamic programming described by Hanson [11, 12] and Naimipour and Hanson [16] which is a modification of the Douglas and Dupont [6] and Douglas [7] Crank–Nicolson, predictor–corrector method for parabolic partial differential equations. In this treatment finite differences are used so that the local truncation error is second order in both the state and time derivatives.

These modifications are for the representation of the Poisson noise terms as well as improvements on the method. The type of Poisson noise used in this model which yield the delay terms here are expressed as integrals and not sums. For this reason, numerical integration must be used that should be of the same order as the method used to solve the differential equation. In this model the regular or optimal control can not be found explicitly as was the case in [11, 16, 12]. This is due to the bilinear coefficient in (1) of the Poisson noise $[H_1(\mathbf{X}(t),t)\mathbf{U}(t)]d\mathbf{P}_1(t)$. This leads to additional complications in terms of determining the regular or optimal control of (18), since there is an implicit relationship between $\mathbf{u}^{\mathrm{reg}}(t)$ and the gradient of $v(\mathbf{x},t)$. The value for the switching term and the optimal or regular control need to be determined numerically, for example by using a *conjugate gradient multidimensional search* (see [17]) or by predictor-corrector iteration.

An LQGP approximate analysis is used in the predictor step to accelerate the convergence of the method. This reduces the number of corrector steps necessary. However, some of the coefficients for the defining SDE (1) and the quadratic performance index (10) must be approximated as an LQGP problem as follows:

$$\begin{aligned}
F_0(\mathbf{x},t) &\to A(t)\mathbf{x}+\mathbf{C}(t);\\
F_1(\mathbf{x},t) &\to B(t);\\
H_0(\mathbf{x},t) &\to [\bar{H}_1(t)\mathbf{x}]+\bar{H}_3(t);\\
H_1(\mathbf{x},t) &\to \bar{H}_2(t);\\
C_0(\mathbf{x},t) &\to \mathbf{x}^T Q(t)\mathbf{x};\\
\mathbf{C_1}(\mathbf{x},t) &\to \mathbf{0}_{n\times 1};\\
C_2(\mathbf{x},t) &\to R(t).
\end{aligned}\tag{23}$$

A convenient way of obtaining these linear approximations is to hold values of the state at reasonable fixed values, for example at the values at t_f . These coefficients can then be used to generate the LQGP approximation for the optimal, expected performance index, $V_{\mathbf{j}k}^{\mathrm{LQGP}}$, by solving the LQGP system of nonlinear differential equations (see [19]).

It is assumed that the prior value for the optimal, expected performance index, $V_{\mathbf{j},k-1}$ is known prior to calculating the new time step k for $k=1$ to K. As a requirement of the Crank–Nicolson procedure used for the time derivative, a number of quantities must be determined at the mid-point of the time step, $T_{k-\frac{1}{2}}$. For each new time step, $k=1$ to K, the algorithm starts with a convergence accelerating *extrapolator (x)* *start:*

$$V^{(x)}_{\mathbf{j},k-\frac{1}{2}} = \begin{cases} \frac{1}{2}(3\cdot V_{\mathbf{j},k-1} - V_{\mathbf{j},k-2}), & k\ge 2 \\ \frac{1}{2}(V_{\mathbf{j},k-1} + V^{\mathrm{LQGP}}_{\mathbf{j},k}), & k=1 \end{cases} ; \tag{24}$$

$$\text{Calculate new } \mathbf{DV}^{(x)}_{\mathbf{j},k-\frac{1}{2}}; DDV^{(x)}_{\mathbf{j},k-\frac{1}{2}}; \mathbf{HV0}^{(x)}_{\mathbf{j},k-\frac{1}{2}}; \mathbf{HV1}^{(x)}_{\mathbf{j},k-\frac{1}{2}}; \mathcal{S}^{(x)}_{\mathbf{j},k-\frac{1}{2}}.$$

In [11, 16, 12], the first time step does not use an accelerating extrapolator start since only the values at t_f are known, therefore the first and most crucial time step may have difficulty converging. In the treatment here this problem has been addressed, using an LQGP start for the first new time value.

The *corrector (xpec) step* determines the value of V at the full time time step T_k:

$$\begin{aligned} V^{(xpec,\gamma)}_{\mathbf{j},k} &= V_{\mathbf{j},k-1} + DT\cdot\Bigg[F0^T_{\mathbf{j},k-\frac{1}{2}}\mathbf{DV}^{(xpece,\gamma-1)}_{\mathbf{j},k-\frac{1}{2}} + \tfrac{1}{2}G_{\mathbf{j},k-\frac{1}{2}}G^T_{\mathbf{j},k-\frac{1}{2}} : DDV^{(xpece,\gamma-1)}_{\mathbf{j},k-\frac{1}{2}} \\ &+ C0_{\mathbf{j},k-\frac{1}{2}} + \mathcal{S}^{(xpece,\gamma-1)}_{\mathbf{j},k-\frac{1}{2}} + \textstyle\sum_{i=1}^{q_0}\lambda_{0,i}\int_{\mathcal{Z}_{0i}}\Big[HV0^{(xpece,\gamma-1)}_{i,\mathbf{j},k-\frac{1}{2}} - V^{(xpece,\gamma-1)}_{\mathbf{j},k-\frac{1}{2}}\Big]\phi_{0,i}(z_i,T_k)dz_i\Bigg], \end{aligned} \tag{25}$$

for $\gamma = 0$ to $\gamma_{\max}$ while the convergence criterion is not satisfied. The *extrapolated predictor (xp) step* is the starting corrector step:

$$V^{(xp)}_{\mathbf{j},k} = V^{(xpec,0)}_{\mathbf{j},k},$$

and the input is the *extrapolation (x) start*:

$$V^{(xpece,-1)}_{\mathbf{j},k-\frac{1}{2}} \equiv V^{(x)}_{\mathbf{j},k-\frac{1}{2}}.$$

The *corrector evaluation (xpece) step* is used to determine the various quantities at the half time step, $T_{k-\frac{1}{2}}$, based on the corrector step $V^{(xpec,\gamma)}_{\mathbf{j},k}$ and previous step value $V_{\mathbf{j},k-1}$ to form the Crank-Nicolson average:

$$\begin{gathered} V^{(xpece,\gamma)}_{\mathbf{j},k-\frac{1}{2}} = \tfrac{1}{2}(V_{\mathbf{j},k-1} + V^{(xpec,\gamma)}_{\mathbf{j},k}); \\ \text{Calculate new } \mathbf{DV}^{(xpece,\gamma)}_{\mathbf{j},k-\frac{1}{2}}; DDV^{(xpece,\gamma)}_{\mathbf{j},k-\frac{1}{2}}; \mathbf{HV0}^{(xpece,\gamma)}_{\mathbf{j},k-\frac{1}{2}}; \mathbf{HV1}^{(xpece,\gamma)}_{\mathbf{j},k-\frac{1}{2}}; \mathcal{S}^{(xpece,\gamma)}_{\mathbf{j},k-\frac{1}{2}}. \end{gathered} \tag{26}$$

Upon convergence the final corrector step is used as the value for $v(\mathbf{X}_{\mathbf{j}}, T_k)$, i.e., we set $V_{\mathbf{j},k} = V^{(xpec,\gamma_{\max})}_{\mathbf{j},k}$.

5 Multistage Manufacturing System Example

Consider a multistage manufacturing system (MMS) that produces a single consumable commodity, that requires a process consisting of k linear stages. For example, the manufacture or assembly of an automobile or a printed circuit board (PCB). The initial or loading stage is where all of the raw materials for all stages are input into the system. The final or unloading stage is the mechanism by which the final product is delivered to the consumer or distributer. For simplicity, the loading and unloading stages are not considered as part of the MMS. This model is similar to a flexible manufacturing system (FMS), but its perspective is global instead of local. Each stage can be viewed as an FMS. Kimemia and Gershwin [14] describe the similarities and differences between FMS and MMS, while presenting an algorithm for FMS control.

The example considered is a modification of the MMS presented by Westman and Hanson [19]. In which a linear dynamical system is employed with the state variables consisting of the number of operational workstations and the surplus aggregate level for each stage, where the production rate was a parameter. In the model presented here, the dynamics are nonlinear and the production rate is a state variable. The cost function used is similar to that of [19] with an additional term used to penalize shortfalls or surplus of production during the planning horizon.

For each stage $i = 1$ to k, at time t into the manufacturing or planning horizon, there are $n_i(t)$ operational workstations which are assumed to have identical properties and produce at the same rate $c_i(t)$ with a capacity of producing M_i parts per unit time. Each workstation is subject to failure and can be repaired with the mean time between failures and the repair duration exponentially distributed. The production rate $c_i(t)$ is the utilization, i.e., the fraction of time busy, of the workstations at stage i. For similar models with variations see Akella and Kumar [1] for a treatment of optimal inventory levels, as well as Boukas and coworkers [3, 4, 13] for a treatment that includes preventive maintenance and machine age structure, and Gershwin [8] for a treatment that includes the production scheduling of several parts.

The state of the manufacturing system is described by the number of operational workstations, $n_i(t)$, the surplus aggregate levels, $a_i(t)$, and by the production rates, $c_i(t)$. The total number of workstations at stage i is N_i. Hence, $n_i(t)$ can be viewed as a birth and death process confined to the interval $0 \leq n_i(t) \leq N_i$ with random jump sizes. The defining equation for the number of operational workstations is given by:

$$dn_i(t) = dP_i^R(t) - dP_i^F(t), \tag{27}$$

where $dP_i^R(t)$ and $dP_i^F(t)$ are Poisson processes used to model the repair (birth) processes and the failure (death) processes, respectively. The surplus aggregate level represents the surplus (if positive) or shortfall (if negative) of the production of pieces that have successfully completed i stages of the manufacturing process. The state equation for the surplus aggregate level for stage $i = 1$ to k is given by:

$$da_i(t) = \left[M_i n_i(t)(c_i(t) + u_i(t)) - d_i(t)\right] dt + g_i(t)dW_i(t). \tag{28}$$

The change in the surplus aggregate level, $da_i(t)$, is determined by the number of pieces that have successfully completed i stages of the manufacturing process $(M_i n_i(t)(c_i(t) + u_i(t))dt)$, that are not defective, and are not consumed by stage $i+1$ $(d_i(t)dt)$, and by the status of the workstations. The control $u_i(t)dt$ is used to adjust the production rate, $c_i(t)$, to compensate for workstation repair and failure and random fluctuations in the manufacturing process. The last term, $g_i(t)dW_i(t)$, is used to model small random fluctuations in the number pieces produced, for example the number of defective pieces. The state equation for the production rate for stage i is given by:

$$dc_i(t) = u_i(t)dt. \tag{29}$$

This equation is new to this model and allows for a dynamical determination of the production rate, which was not the case in [19].

The demand rate $d_i(t)$ is the number of pieces needed per unit time to insure that the manufacturing process is a continuous flow of work, so that the desired number of completed pieces are produced. The demand rate must also take into account, based on past history, a minimal buffer level sufficient to compensate for defective pieces as well as workstation failures, and to insure that the proper start-up levels are present for the next planning horizon. The demand is bounded by 0 and the maximum production rate $M_i N_i$.

The outputs of the system (28, 27, 29) are various measures for the production rate provided that $n_i(t) \neq 0$ and the expected value for the surplus aggregate level at the end of the planning horizon, $\hat{a}_i(t)$. The virtual production rate, $c_i^{\mathrm{vir}}(t)$, does not anticipate workstation repair or failure or compensate for random fluctuations in the number of pieces produced, is defined as:

$$c_i^{\mathrm{vir}}(t) \equiv \frac{\int_t^{t_f} d_i(\tau)d\tau}{M_i n_i(t)}. \tag{30}$$

The regular controlled production rate, $c_i^{\mathrm{reg}}(t)$, anticipates workstation repair and failure as well as fluctuations in the number of pieces produced, and is given by:

$$c_i^{\mathrm{reg}}(t) = c_i(t_f) + \int_t^{t_f} u_i^{\mathrm{reg}}(\tau)d\tau. \tag{31}$$

It is based on the regular or unconstrained control $u_i^{\mathrm{reg}}(t)$ and therefore may not be physically realizable, i.e. not admissible. The optimal control, $u_i^*(t)$, is constrained so that the constrained controlled production rate, $c_i^*(t)$, is admissible. It is determined by:

$$c_i^*(t) = c_i(t_f) + \int_t^{t_f} u_i^*(\tau)d\tau, \tag{32}$$

and is bounded by:

$$0 \leq c_i^*(t) \leq c_{\max,i}(t), \tag{33}$$

where

$$c_{\max,i}(t) \equiv \text{Min}\left[1, \frac{c_{i-1}(t) * M_{i-1} * n_{i-1}(t)}{M_i * n_i(t)}\right], \tag{34}$$

due to the physical and production limitations, respectively. If $n_i(t) = 0$, then all production rates and controls for stage i are set to 0. The expected value for the surplus aggregate level at the end of the planning horizon, $\hat{a}_i(t)$, measures the deviation from the target production goal based on the current status of the manufacturing system, is determined by:

$$\hat{a}_i(t) = a_i(t_f) + \begin{cases} M_i n_i(t)(c_i^*(\tau) - c_i^{\text{reg}}(\tau)(t_f - t), & \text{for } n_i(t) \neq 0 \\ \int_t^{t_f} d_i(\tau)d\tau, & \text{for } n_i(t) = 0 \end{cases}. \tag{35}$$

The cost functional to be minimized is a quadratic cost-to-go form given by (10) with $C_0(\mathbf{X}, t) \equiv \mathbf{X}^T \widehat{C}_0(t)\mathbf{X}$ and $\mathbf{C_1}(\mathbf{X}, t) \equiv \mathbf{0}_{k\times 1}$ where the state or plant vector is given by: $\mathbf{X}(t) = [\mathbf{a}(t) \quad \mathbf{n}(t) \quad \mathbf{c}(t)]^T$. The salvage cost matrix S is a symmetric, positive semi-definite $3k \times 3k$ matrix. The salvage cost is based on the idea of *Just In Time* or *stockless production* (see Hall [10]). This means that a penalty is imposed if we have a surplus or shortfall of production goal at the end of the planning horizon in any stage of the manufacturing system. Further motivation for the salvage term is given by Bielecki and Kumar [2], who show that, for an unreliable manufacturing system, the optimal policy is a zero inventory policy. The integral term of (10), $\frac{1}{2}\mathbf{U}^T(t)C_2(\mathbf{X}, t)\mathbf{U}(t)$, is used to force a minimum effort discipline. This cost functional is different than the one in [19] since an additional penalty is imposed for shortfalls or surplus of production during the planning horizon. Motivation for this term would be the cost of storing additional pieces or loss due to reduced production.

For numerical concreteness, consider a MMS with $k = 3$ stages with a planning horizon of 40 hours. Let the initial surplus aggregate level for all stages be zero. Let the demand be $d_i(t) = 285$ pieces per hours for $i = 1$ to 3. The Gaussian random fluctuations of production will be assumed absent, i.e., $g_i(t) = 0$ for $i = 1$ to 3. Let the total number of workstations, $N_i(t)$, for each stage be 3, 5, and 4, respectively. The salvage cost and the instantaneous quadratic cost coefficient matrices are given by:

$$S(t_f) = \begin{bmatrix} S_1(t_f) & 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}, \quad S_1(t_f) = \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 1.9 & 0 \\ 0 & 0 & 2.6 \end{bmatrix}, \tag{36}$$

$$\widehat{C}_0(t) = \begin{bmatrix} C_{0,1}(t) & 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}, \quad C_{0,1}(t) = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1.1 & 0 \\ 0 & 0 & 1.9 \end{bmatrix}, \tag{37}$$

and

$$C_2(\mathbf{X}(t), t) = \begin{bmatrix} 1.3E4(M_1 n_1(t))^2 & 0 & 0 \\ 0 & 0.8E4(M_2 n_2(t))^2 & 0 \\ 0 & 0 & 0.6E4(M_3 n_3(t))^2 \end{bmatrix}. \tag{38}$$

The individual characteristics for an individual given workstation is summarized in the table below.

Stage i	Production Capacity, M_i (pieces/hour)	Mean Time between Failure $1/\lambda_i^F$ (hours)	Mean Time to Repair $1/\lambda_i^R$ (hours)
1	117	25.0	2.5
2	71	21.5	4.0
3	88	30.0	1.5

By comparing equations (27, 28, and 29) to the defining SDE (1) the coefficients are given by:

$$F_0(\mathbf{X}(\mathbf{t}),t) = \begin{bmatrix} [M_i c_i(t) n_i(t) - d_i(t)]_{3\times 1} \\ \mathbf{0}_{3\times 1} \\ \mathbf{0}_{3\times 1} \end{bmatrix}, \quad F_1(\mathbf{X}(\mathbf{t}),t) = \begin{bmatrix} \mathrm{diag}[\mathbf{M}]\mathrm{diag}[\mathbf{n}(t)] \\ \mathbf{0}_{3\times 3} \\ I_{3\times 3} \end{bmatrix},$$

$$H_0(t) = \begin{bmatrix} 0_{3\times 3} & 0_{3\times 3} \\ I_{3\times 3} & -I_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}, \qquad d\mathbf{P}_0(t) = \begin{bmatrix} d\mathbf{P}^R(t) \\ d\mathbf{P}^F(t) \end{bmatrix}, \tag{39}$$

$$G(t) = 0, \qquad H_1(\mathbf{X}(\mathbf{t}),t),t) = 0, \qquad d\mathbf{P}_1(t) = \mathbf{0}$$

with $\mathrm{diag}[\mathbf{c}(t)] = [\Sigma_j \delta_{i,j} c_j]_{k\times k}$ being the diagonalization of the vector whose components are given by $c_i(t)$ for $i = 1$ to k. The corresponding coefficients for the LQGP analysis is given by:

$$A(t) = \begin{bmatrix} 0_{3\times 3} & \mathrm{diag}[\mathbf{M}]\mathrm{diag}[\mathbf{c}(t_f)] & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}, \quad B(t) = \begin{bmatrix} \mathrm{diag}[\mathbf{M}]\mathrm{diag}[\mathbf{n}(t)] \\ \mathbf{0}_{3\times 3} I_{3\times 3} \end{bmatrix}, \quad \mathbf{C}(t) = \begin{bmatrix} -\mathbf{d}(t) \\ \mathbf{0}_{3\times 1} \end{bmatrix},$$

$$H_1(t) = 0, \qquad H_2(t) = 0, \qquad H_3(t) = \begin{bmatrix} 0_{3\times 3} & 0_{3\times 3} \\ I_{3\times 3} & -I_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{bmatrix},$$

$$d\mathbf{P}_1(t) = \mathbf{0}, \qquad d\mathbf{P}_2(t) = \mathbf{0}, \qquad d\mathbf{P}_3(t) = \begin{bmatrix} d\mathbf{P}^R(t) \\ d\mathbf{P}^F(t) \end{bmatrix}, \tag{40}$$

$$Q(t) = \begin{bmatrix} Q_1(t) & 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} \end{bmatrix}, \quad Q_1(t) = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1.1 & 0 \\ 0 & 0 & 1.9 \end{bmatrix},$$

$$R(t) = \begin{bmatrix} 1.3E4(M_1 n_1(t_f))^2 & 0 & 0 \\ 0 & 0.8E4(M_2 n_2(t_f))^2 & 0 \\ 0 & 0 & 0.6E4(M_3 n_3(t_f))^2 \end{bmatrix}.$$

The results are illustrated in the following figures for the second stage in the manufacturing process with [3,4,*] active workstations. In Figure 1, the regular and constrained controlled production rates, $c_2(t)$, exhibit the anticipation of workstation repair and failure, whereas the maximal, virtual production rate does not. The constrained controlled production rate saturates at 100% of capacity, since the unconstrained, regular controlled production rate exceeds 100% of capacity which is not physically realizable. In Figure 2, the expected value for the surplus aggregate level, $\hat{a}_2(t)$, becomes negative since the anticipated demand is not being met. These figures display some preliminary results. Due to large computational time and memory use, complete planning horizon results are not currently available. Complete results will be available for the time of publication. Progress on the numerical results will be available on the first author's homepage [http://www.math.uic.edu/~jwestman/].

6 Conclusions

These results are of great interest because the NLQGP problem is a *nonlinear* theoretical model and a canonical computational model. The Poisson random components permit the model to account for a wide

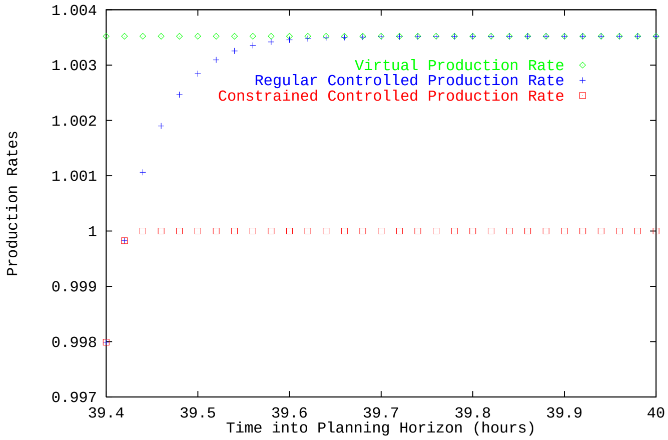

Figure 1: Virtual, regular and constrained controlled production rates, $c_2(t)$, for stage 2 with [3,4,*] active workstations.

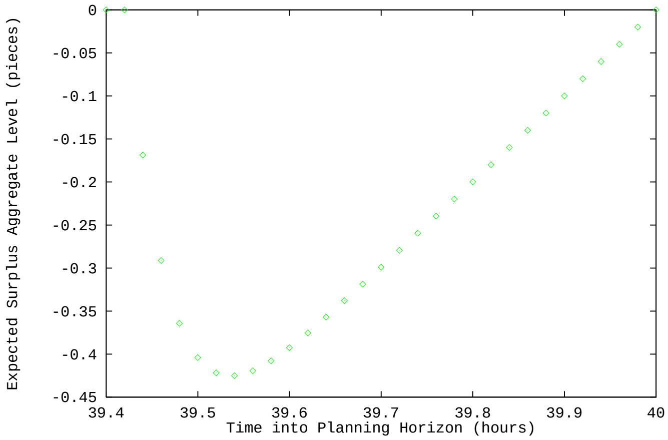

Figure 2: The expected value for the surplus aggregate level, $hata_2(t)$, for stage 2 with [3,4,*] active workstations.

variety of large fluctuations biased by the values of the state and control, whereas Gaussian noise is useful for relatively milder background fluctuations. The stochastic differential equation formulation used here makes the construction of models analogous to that for other dynamic systems. These features of this model allow for greater realism with out having to compromise aspects of physical reality than does the corresponding linear model. This problem formulation would be useful for investigating the response of systems subject to extreme random inputs or catastrophic failure via computer experiments, as well as many other applications. The multistage manufacturing system (MMS) application is a good test of our methods, since it is a nonlinear model with Poisson modeled catastrophic failures of the workstations subject to background fluctuations in production.

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