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Computational Method for Nonlinear Stochastic Optimal Control *

J. J. Westman
Department of Mathematics
University of California
Box 951555
Los Angeles, CA 90095-1555, USA
e-mail: jwestman@math.ucla.edu
URL: http://www.math.ucla.edu/~jwestman/
and
F. B. Hanson
Laboratory for Advanced Computing
Department of Mathematics, Statistics, and Computer Science
University of Illinois at Chicago
851 Morgan St.; M/C 249
Chicago, IL 60607-7045, USA
e-mail: hanson@math.uic.edu
URL: http://www.math.uic.edu/~hanson/

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Abstract

Nonlinear stochastic optimal control problems are treated that are nonlinear in the state dynamics, but are linear in the control. The cost functional is a general function of the state, but the costs are quadratic in the control. The system is subject random fluctuations due to discontinuous Poisson noise that depends on both the state and control, as well as due to discontinuous Gaussian noise. This general framework provides a comprehensive model for numerous applications that are subject to random environments. A stochastic dynamic programming approach is used and the theory for an iterative algorithm is formulated utilizing a least squares equivalent of a genuine LQGP problem to approximate the nonlinear state space dependence of the LQGP problem in control only in order to accelerate the convergence of the nonlinear control problem. A particular contribution of this paper is the treatment of a Poisson jump process that is linear in the control vector, within the contex of a nonlinear control problem.

1. Introduction

A nonlinear stochastic optimal control problem is considered that is a Linear Quadratic Gaussian Poisson problem in control only (i.e., LQGP/U). This problem has nonlinear state dynamics, but has linear control dynamics, while having Gaussian and Poisson random perturbations. The dynamics are governed by a stochastic differential equation (SDE), (1), which describes fairly general Markov processes in continuous time.

The LQGP/U problem generalizes the model of the LQGP problem (see [15, 11]) which is an extension of the LQG analysis (see [1, 8, 2]). In contrast to the LQGP problem, the LQGP/U problem does not exhibit,

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in general, a formal closed form solution. Numerical methods for partial differential equations are necessary in order to solve the Hamilton-Jacobi-Bellman (HJB) equation of stochastic dynamic programming for the LQGP/U problem, but the LQGP/U does not lead to a simple direct representation of the optimal control in terms of the state space. However, like stochastic dynamic programming, the computational requirements for the LQGP/U problem, with a search over all the state space, are much greater than that for a pure LQGP problem.

The LQGP/U problem extends the *LQGP in control only* model used by Hanson [5, 6], and Naimipour and Hanson [9], by utilizing a more complex and realistic form for the Poisson processes. In this paper, we use a *marked Poisson process* so that the jump amplitudes or marks are random variables with an associated probability function that is independent of the arrival process. In other words, a marked Poisson process represents a sequence of ordered pairs $(T_1, M_1), \quad (T_2, M_2), \quad \ldots, \quad (T_k, M_k)$, in which T_i is the time of occurrence of the ith jump with amplitude M_i (see, for example [10]). This representation of the Poisson process provides more realism and flexibility for a wider range of stochastic control applications. The high performance techniques developed by Hanson et al. [5, 6, 9] for handling the large computational demands of the PDE of stochastic dynamic programming for Markovian problems in continuous time can be used for the marked Poisson process that we use here.

Discrete jumps of random amplitude occur in the state because of the Poisson terms, which are superimposed on the continuous, yet nonsmooth Gaussian noise contribution. This combination of discrete and continuous system is called a *hybrid system*. Random background fluctuations are modeled by Gaussian noise processes. Poisson noise is useful for modeling catastrophic events, for example machine failure (see [11, 12, 13]). Poisson SDEs have also been used to model both natural disasters and bonanzas in population growth (see [7]). In this presentation, the analysis should go through for the more general martingales.

The algorithm presented here, advances the previous work of Westman and Hanson [12], by use of a least squares approximation for a LQGP analysis of the nonlinear dynamics in conjunction with a general algorithm to solve simultaneously for the optimal expected cost and optimal control *for the nonlinear problem*. The least squares approximation is used in conjunction with known or computed information from previous time steps to form a *predicted starting value* for the iterative algorithm. The cost of forming this predicted starting value is far less than that for a single search over the discretized state space. This predicted starting value should accelerate the convergence of the algorithm. Additionally, in the previous work of Westman and Hanson [11, 12, 13], the regular or unconstrained control was considered, but in this paper the focus is on a general constraints for the control.

The paper is arranged as follows. In Section 2., the defining SDE and quadratic performance index is presented, a mathematical description is given for the Poisson and Gaussian noise processes with the fundamental infinitesimal conditional moments for the state which are necessary for modeling, as well as the stochastic dynamic programming formulation for the control problem. In Section 3., the formal general theory of new modifications for an accelerating least squares LQGP approximation and for treating multidimensional distributed Poisson jump integrals is described, with application to the predictor-corrector Crank-Nicolson method for solving approximation of the Hamilton-Jacobi-Bellman equation.

2. Problem Formulation

The governing stochastic differential equation (SDE) that defines the dynamical system is given by:

$$\begin{aligned} d\mathbf{X}(t) &= [\mathbf{F}_0(\mathbf{X}(t),t) + F_1(\mathbf{X}(t),t)\mathbf{U}(t)]dt + G_0(\mathbf{X}(t),t)d\mathbf{W}(t) \\ &+ H_0(\mathbf{X}(t),t)d\mathbf{P}_0(t) + [H_1(\mathbf{X}(t),t)\mathbf{U}(t)]d\mathbf{P}_1(t), \end{aligned} \tag{1}$$

which represents the evolution of the state influenced the control subject to a random environment. Here, $\mathbf{X}(t)$ is the $m \times 1$ state vector in the state space $\mathcal{D}_{\mathbf{x}}$, $\mathbf{U}(t)$ is the $n \times 1$ control vector in the control space $\mathcal{D}_{\mathbf{u}}$, $d\mathbf{W}(t)$ is the $r \times 1$ Gaussian noise vector, and $d\mathbf{P}_\ell(t)$ is the $q_\ell \times 1$ space-time Poisson noise vector for $\ell = 0$ or 1. The dynamic coefficients (subscript "0" denotes control-independence and subscript "1" denotes linear in control) are the following: $\mathbf{F}_0(\mathbf{x},t)$ is the $m \times 1$ control-independent part of the drift, $F_1(\mathbf{x},t)$ is the $m \times n$ coefficient of the linear control term in the drift, $G_0(\mathbf{x},t)$ is the $m \times r$ amplitude of the Gaussian noise, and $H_0(\mathbf{x},t)$ is the $m \times q_0$ jump amplitude of the control independent Poisson noise. The linear coefficient

$H_1(\mathbf{x},t)$ is the nonstandard linear algebra form, involving the *array valued right-sided dot product*,

$$[H_1(\mathbf{x}(t),t)\mathbf{u}(t)] = [\sum_{k=1}^{n} H_{1,i,j,k}(\mathbf{x},t)\mathbf{u}_k(t)]_{m\times q_1}, \tag{2}$$

which is the control dependent jump amplitude of the Poisson noise, and is a particular feature of the contributions of this paper. The problem formulation given here is the same as that for [12], and similar to that of [13] which does not include the term $[H_1(\mathbf{X}(t),t)\mathbf{U}(t)]d\mathbf{P}_1(t)$. However, in this paper we consider general control constraints as opposed to the regular control which was the case in [12] and a formalization of the method is given based on the idea originally appearing in [12]. A term could be included for a Gaussian noise which is linear in the control, but this term is omitted here for simplicity. The SDE (1) is interpreted in the sense of Itô (see [4], or see [6] for an applicable exposition).

The properties of the Poisson and Gaussian stochastic noise processes are presented, since they are useful for building application models. The Gaussian noise term, $d\mathbf{W}(t)$, consists of r independent, standard Wiener processes $dW_i(t)$, for $i = 1$ to r. The Gaussian components are assumed to have zero infinitesimal mean and diagonal covariance,

$$\text{Mean}[d\mathbf{W}(t)] = \mathbf{0_{r\times 1}} \qquad \text{and} \qquad \text{Covar}[d\mathbf{W}(t), d\mathbf{W}^\top(t)] = I_r dt. \tag{3}$$

The space-time Poisson noise terms, $d\mathbf{P}_l(t) = [dP_{l,i}(t)]_{q_l\times 1}$, consist of q_l independent differentials of space-time Poisson processes for each $l = 0$ or 1, related to the Poisson random measure, $\mathcal{P}_{l,i}(dz_i, dt)$ (see Gihman and Skorohod [4]):

$$dP_{l,i}(t) = \int_{\mathcal{Z}_{l,i}} z_i \mathcal{P}_{l,i}(dz_i, dt), \tag{4}$$

where z_i is the Poisson jump amplitude random variable or the mark of the $dP_{l,i}(t)$ Poisson process for $l = 0$ or 1 where $i = 1$ to q_l, with mean or expectation:

$$\text{Mean}[d\mathbf{P}_l(t)] = \Lambda_l(t)dt \int_{\mathcal{Z}_l} \mathbf{z}\phi_l(\mathbf{z},t)d\mathbf{z} = \Lambda_l(t)dt \left[\int_{\mathcal{Z}_{l,i}} z_i \phi_{l,i} dz_i\right]_{q_l\times 1} \equiv \Lambda_l \overline{\mathbf{Z}}_l dt, \tag{5}$$

where $\Lambda_l(t)$ is the diagonal matrix representation of the Poisson rates $\lambda_{l,i}(t)$ for $i = 1$ to q_l, $\overline{\mathbf{Z}}_l(t)$ is the mean of the jump amplitude mark vector and $\phi_{l,i}(z_i,t)$ is the density of the (l,i)th amplitude mark component. Assuming component-wise independence, $d\mathbf{P}_l(t)$ has covariance given by:

$$\text{Covar}[d\mathbf{P}_l(t), d\mathbf{P}_l^\top(t)] = \Lambda_l(t)dt \int_{\mathcal{Z}_l} (\mathbf{z}-\overline{\mathbf{Z}})(\mathbf{z}-\overline{\mathbf{Z}})^\top \phi_l(\mathbf{z},t)d\mathbf{z} \equiv \Lambda_l(t)\sigma_l(t)dt = [\lambda_{l,i}\sigma_{l,i,j}\delta_{i,j}]_{q_l\times q_l} dt, \tag{6}$$

with $\sigma_l(t) = [\sigma_{l,i,j}\delta_{i,j}]_{q_l\times q_l}$ denoting the diagonalized covariance of the amplitude mark distribution for $d\mathbf{P}_l(t)$. Note that the mark vector is not assumed to have a zero mean, i.e., $\overline{\mathbf{Z}} \neq 0$, permitting some additional modeling complexity.

The Gaussian and Poisson noise processes are assumed to be independent, i.e.,

$$\text{Covar}[d\mathbf{W}(t), d\mathbf{P}_l^\top(t)] = 0_{r\times q_l}. \tag{7}$$

The assumptions of diagonal covariances (3, 6) and independence of the Gaussian and Poisson noise processes (7) leads to a simpler forms in terms of implementation of the algorithm, but do not affect the analysis in the following sections.

The jth jump of the $\{l,i\}$th space-time Poisson process at time $t_{l,i,j}$ with amplitude $z_{l,i,j}$ causes a discrete jump from $t^-_{l,i,j}$ to $t^+_{l,i,j}$ in the state whose amplitude is given by the following expression

$$[\mathbf{X}](t_{l,i,j}) = \left\{ \begin{array}{ll} [[H_0(\mathbf{X}(t^-_{l,i,j}),t_{l,i,j})]_{i,\ell} z_{l,i,j}]_{m\times 1}, & l=0 \\ [[H_1(\mathbf{X}(t^-_{l,i,j}),t_{l,i,j})\mathbf{U}(t^-_{l,i,j})]_{i,\ell} z_{l,i,j}]_{m\times 1}, & l=1 \end{array} \right\}, \text{ for } i=1 \text{ to } q_l \text{ and all } j, \tag{8}$$

i.e., depending on the Poisson jump process l and component i, as well as the jump count j.

The above statistical properties of the stochastic processes, $d\mathbf{W}$ and $d\mathbf{P}_l$, imply that the conditional infinitesimal expectation of the state is

$$\begin{aligned} &\text{Mean}[d\mathbf{X}(t) \mid \mathbf{X}(t) = \mathbf{x}, \mathbf{U}(t) = \mathbf{u}] \\ &\quad = [F_0(\mathbf{x}, t) + F_1(\mathbf{x}, t)\mathbf{u} + H_0(\mathbf{x}, t)\Lambda_0\overline{\mathbf{Z}}_0(t) + [H_1(\mathbf{x}, t)\mathbf{u}]\Lambda_1\overline{\mathbf{Z}}_1(t)]dt \end{aligned} \tag{9}$$

and the conditional infinitesimal covariance,

$$\begin{aligned} &\text{Covar}[d\mathbf{X}(t)d\mathbf{X}^\top(t) \mid \mathbf{X}(t) = \mathbf{x}, \mathbf{U}(t) = \mathbf{u}] \\ &= \left[(G_0G_0^T)(t) + H_0(\mathbf{x}, t)(\Lambda_0\sigma_0)(t)H_0^T(\mathbf{x}, t) + [H_1(\mathbf{x}, t)\mathbf{u}](\Lambda_1\sigma_1)(t)[H_1(\mathbf{x}, t)\mathbf{u}]^T\right] dt. \end{aligned} \tag{10}$$

The conditional infinitesimal moments (9) and (10) as well as the description of the discrete jumps introduced into the state (8) are fundamental for modeling applications. Note that the conditional infinitesimal mean (9) is determined by the nonlinear plant function and the nonzero mean of the jump distributed Poisson noise, whereas the conditional infinitesimal covariance (10) is determined exclusively from covariance of the Gaussian and Poisson noise processes compounded by their respective amplitude functions, G_0, H_0, and $[H_1\mathbf{u}]$.

As with the usual LQG problem, the quadratic (Q) part of the LQGP/U denotes the quadratic performance index with respect to control costs and permitting the costs of the state dynamics to be generally nonlinear. The performance index is a measure of cost that the manager of a plant seeks to optimize. Here, the quadratic performance index is the cost-to-go form:

$$V(\mathbf{X}, \mathbf{U}, t) = \frac{1}{2}\mathbf{X}^\top(t_f)S_f\mathbf{X}(t_f) + \int_t^{t_f} C(\mathbf{X}(\tau), \mathbf{U}(\tau), \tau)d\tau, \tag{11}$$

where the time horizon is $[t, t_f)$, the instantaneous quadratic cost function in control is

$$C(\mathbf{x}, \mathbf{u}, t) = C_0(\mathbf{x}, t) + \mathbf{C}_1^\top(\mathbf{x}, t)\mathbf{u} + \frac{1}{2}\mathbf{u}^\top C_2(\mathbf{x}, t)\mathbf{u}, \tag{12}$$

and S_f is the final cost matrix. This final cost is given by the quadratic form, $\mathbf{x}^\top S_f\mathbf{x} = S_f : \mathbf{x}\mathbf{x}^\top = \text{Trace}[S_f\mathbf{x}\mathbf{x}^\top]$. The quadratic cost coefficient C_2 is assumed to be a symmetric, positive definite $n \times n$ array, while S_f is assumed to be symmetric, positive semi-definite $m \times m$ array in order to minimize (11). The linear cost coefficient $\mathbf{C}_1$ is a $n \times 1$ vector and C_0 is a scalar. The quadratic performance index (11,12) was selected to be the most general form for the LQGP/U problem.

The analysis contained in this paper allows for a more general form of the performance index (11,12), with few modifications necessary. However, for simplicity the quadratic form is used to correspond with our previous work [11, 12, 13]. In the absence of control constraints the use of the quadratic performance index described here, admits a unique regular control. The use of a more general performance index and/or general control constraints may not necessarily admit a unique optimal control, and thus would require additional support or requirements for the model to be well posed.

The optimal, expected performance, $v(\mathbf{x}, t)$, is

$$v(\mathbf{x}, t) \equiv \min_{\mathbf{u}[t,t_f)} \left[\underset{\mathbf{P},\mathbf{W}[t,t_f)}{\text{Mean}} [V(\mathbf{X}, \mathbf{U}, t) \mid \mathbf{X}(t) = \mathbf{x}, \mathbf{U}(t) = \mathbf{u}] \right]. \tag{13}$$

The state and control belong to the admissible classes for the state, $\mathcal{D}_\mathbf{x}$, and control, $\mathcal{D}_\mathbf{u}$, respectively. The final condition for the optimal, expected value is determined by

$$v(\mathbf{x}, t_f) = \frac{1}{2}\mathbf{x}^\top S_f\mathbf{x} \quad \text{for} \quad \mathbf{x} \in \mathcal{D}_\mathbf{x}. \tag{14}$$

Stochastic dynamic programming is an application of the principle of optimality to the optimal expected performance value (13,11,12), and the general Itô chain rule for Markov stochastic processes in continuous time (see [4], or see [6] for a more tractable derivation). This chain rule differs from the chain rule of calculus

in that it must account for the lack of continuity for the Poisson noise or lack of smoothness for Gaussian noise, and not too much on the randomness of the noise. The form of the chain rule needed here is

$$\begin{aligned} dv(\mathbf{X}(t),t) &= v(\mathbf{X}(t+dt),t+dt) - v(\mathbf{X}(t),t) \\ &= \left(\frac{\partial v}{\partial t} + (\mathbf{F}_0 + F_1\mathbf{U}(t))^\top \nabla_x[v] + \frac{1}{2}(G_0 G_0^\top) : \nabla_x[\nabla_x^\top[v]]\right)(\mathbf{X}(t),t)dt \\ &+ \left((G_0 d\mathbf{W}(t))^\top \nabla_x[v]\right)(\mathbf{X}(t),t) \\ &+ \int_{\mathcal{Z}_0} [v(\mathbf{X}(t) + H_0(\mathbf{X}(t),t)\mathbf{z},t) - v(\mathbf{X}(t),t)]\,\mathcal{P}_0(d\mathbf{z},dt) \\ &+ \int_{\mathcal{Z}_1} [v(\mathbf{X}(t) + [H_1(\mathbf{X}(t),t)\mathbf{U}(t)]\mathbf{z},t) - v(\mathbf{X}(t),t)]\,\mathcal{P}_1(d\mathbf{z},dt), \end{aligned} \tag{15}$$

where the double dot product is defined by $A : B = \sum_i \sum_j A_{i,j}B_{j,i} = \text{Trace}[AB]$ and $\nabla_x[v]$ is the gradient in the state vector. The integral terms of (15) represents the total contributions of the jumps in the state from the Poisson processes. Using the chain rule, along with the principle of optimality over the time horizon, yields the partial differential equation of stochastic dynamic programming:

$$\begin{aligned} 0 &= \frac{\partial v}{\partial t}(\mathbf{x},t) + \underset{\mathbf{u}\in\mathcal{D}_\mathbf{u}}{\text{Min}}\Big[(\mathbf{F}_0(\mathbf{x},t) + F_1(\mathbf{x},t)\mathbf{u})^\top \nabla_x[v](\mathbf{x},t) \\ &+ \frac{1}{2}\left(\left(G_0 G_0^\top\right) : \nabla_x[\nabla_x^\top[v]]\right)(\mathbf{x},t) + C_0(\mathbf{x},t) + \mathbf{C}^\top(\mathbf{x},t)\mathbf{u} + \frac{1}{2}\mathbf{u}^\top C_2(\mathbf{x},t)\mathbf{u} \\ &+ \sum_{l=1}^{q_0} \lambda_{0,l} \int_{\mathcal{Z}_{0,l}} [v(\mathbf{x} + \mathbf{H}_{0,l}(\mathbf{x},t)z_l,t) - v(\mathbf{x},t)]\,\phi_{0,l}(z_l,t)dz_l \\ &+ \sum_{l=1}^{q_1} \lambda_{1,l} \int_{\mathcal{Z}_{1,l}} [v(\mathbf{x} + [H_1(\mathbf{x},t)\mathbf{u}]_{il}z_l,t) - v(\mathbf{x},t)]\,\phi_{1,l}(z_l,t)dz_l\Big], \end{aligned} \tag{16}$$

where the lth component of the vector jump coefficients are given by $\mathbf{H}_{0,l}(\mathbf{x},t) \equiv [H_{0,i,l}(\mathbf{x},t)]_{m\times 1}$ and $[H_1(\mathbf{x},t)\mathbf{u}]_{il} \equiv [[H_1(\mathbf{x},t)\mathbf{u}]_{il}]_{m\times 1}$ for $l = 1$ to q. The backward functional partial differential equation (PDE) (16) is known as the Hamilton-Jacobi-Bellman (HJB) equation and is subject to the final condition equation (14). The argument of the minimum function of equation (16) is called the Hamiltonian of the system. Rewriting the HJB equation (16) into control independent and dependent parts yields:

$$\begin{aligned} 0 &= \frac{\partial v}{\partial t}(\mathbf{x},t) + C_0(\mathbf{x},t) + \left(\mathbf{F}_0{}^\top \nabla_x[v]\right)(\mathbf{x},t) + \frac{1}{2}\left(\left(G_0 G_0^\top\right) : \nabla_x[\nabla_x^\top[v]]\right)(\mathbf{x},t) \\ &+ \sum_{l=1}^{q_0} \lambda_{0,l} \int_{\mathcal{Z}_{0,l}} [v(\mathbf{x} + \mathbf{H}_{0,l}(\mathbf{x},t)z_l,t) - v(\mathbf{x},t)]\,\phi_{0,l}(z_l,t)dz_l + \mathcal{S}^*(\mathbf{x},t), \end{aligned} \tag{17}$$

where the control minimization terms have been collected in the control switching term,

$$\begin{aligned} \mathcal{S}^*(\mathbf{x},t) &\equiv \underset{\mathbf{u}\in\mathcal{D}_\mathbf{u}}{\text{Min}}\Big[\mathbf{C}_1^\top(\mathbf{x},t)\mathbf{u} + (F_1(\mathbf{x},t)\mathbf{u})^\top \nabla_x[v](\mathbf{x},t) + \frac{1}{2}\mathbf{u}^\top C_2(\mathbf{x},t)\mathbf{u} \\ &+ \sum_{l=1}^{q_1} \lambda_{1,l} \int_{\mathcal{Z}_{1,l}} [v(\mathbf{x} + [H_1(\mathbf{x},t)\mathbf{u}]_l z_l,t) - v(\mathbf{x},t)]\,\phi_{1,l}(z_l,t)dz_l\Big]. \end{aligned} \tag{18}$$

The argument of the minimum is the optimal control, $\mathbf{u}^*(\mathbf{x},t)$; if there are no control constraints the optimal control is known as the regular control, $\mathbf{u}_{\text{reg}}(\mathbf{x},t)$, which can not be determined explicitly by standard calculus optimization in this case.

An example of a control domain is the use of component-wise or hypercube control constraints, the control domain $\mathcal{D}_\mathbf{u}$ is specified by

$$U_{\min,i}(t) \le u_i \le U_{\max,i}(t), \quad \text{for} \quad i = 1 \quad \text{to} \quad n, \tag{19}$$

and the stochastic optimal control vector $\mathbf{u}^*(\mathbf{x},t)$ are given in terms of components:

$$u_i^\star(\mathbf{x},t) = \left\{ \begin{array}{cc} U_{\min,i}(t), & u_{\mathrm{reg},i}(\mathbf{x},t) < U_{\min,i}(t) \\ u_{\mathrm{reg},i}(t), & U_{\min,i}(t) \le u_{\mathrm{reg},i}(\mathbf{x},t) \le U_{\max,i}(t) \\ U_{\max,i}(t), & U_{\max,i}(t) < u_{\mathrm{reg},i}(\mathbf{x},t) \end{array} \right\}, \quad \text{for} \quad i = 1 \quad \text{to} \quad n. \tag{20}$$

The numerical method presented in the following section can accommodate a more general control domain.

3. Least Squares Approximation for the LQGP/U Problem

Unlike standard PDEs, solving the HJB equation (17,18) with final condition (14) requires finding the optimal value function $v(\mathbf{x},t)$ simultaneously with the optimal control vector $\mathbf{u}^*(\mathbf{x},\mathrm{t})$ and is far too complex to consider exact analytical solutions. Therefore, numerical methods for an approximation for the optimal control and optimal expected performance is necessary. The HJB equation (17) is different from standard PDEs due to its functional dependence embodied in the switching term (18) which is required to get the optimal control along with the optimal value, and the the global dependence of the Poisson jump integral term is unlike the local dependence of standard partial derivatives. This global functional dependence makes Poisson noise terms far more complex than the local second order partial derivatives due to Gaussian noise.

3.1. Least Squares Approximation for LQGP/U Coefficients

We present the results for the *Least Squares Approximation* (LSA) for the coefficients of a nonlinear optimal control problem that is an LQGP problem in the control n-vector $\mathbf{u}$ only (i.e., LQGP/U), but can be arbitrarily nonlinear in the state m-vector $\mathbf{x}$ from [13] along with the extensions necessary to accommodate the bilinear term for the control and Poisson noise, $[H_1(\mathbf{X}(t),t)\mathbf{U}(t)]d\mathbf{P}_1(t)$. In LSA, the approximated coefficients define a true LQGP problem that is LQGP in both control and state vectors. The motivation is that the least squares approximate LQGP problem is a reasonable estimate of an averaged nonlinear problem. In this presentation, we do not assume a regular control but a general constrained control. This adds additional complexity not only to the control model but the corresponding numerical solution as well, since we need to determine the value of the switching term (27). Using the LSA to generate the associated LQGP in state and control, the state-space search necessary for the nonlinear dynamical case is replaced by a solution of a system of coupled ordinary differential equations in time only (34,35,37), while the state dependence of the LSA optimal value can be recovered directly from the approximate LQGP quadratic form in state space (24). Therefore, the computational demands for the LSA-LQGP approximation to the nonlinear control problem is far less than a single search over the discretized state–control space. The primary target of LSA is the first time step where there is no second value to extrapolate from the final condition (14).

The LSA for the LQGP/U problem (denoted here by superscript (0)) approximates the coefficients of (1,12) so that they are valid for the genuine LQGP problem. The coefficients can be of three types: state independent, linear in the state, or quadratic in the state. Time, t, is merely a parameter held fixed in the least squares procedure. Listed below are the LQGP forms of each approximate state-dependent coefficient:

- Linear Control-Independent Dynamics Coefficient ($m \times 1$):

$$\mathbf{F}_0^{(0)}(\mathbf{x},t) = \mathbf{F}_{0,0}^{(0)}(t) + F_{0,1}^{(0)}(t)\mathbf{x};$$

- State-Independent Linear Control Dynamics Coefficient ($m \times n$):

$$F_1^{(0)}(\mathbf{x},t) = F_{1,0}^{(0)}(t);$$

- Control-Independent, State-Dependent Gaussian Coefficient ($m \times r$):

$$G_0^{(0)}(\mathbf{x},t) = G_{0,0}^{(0)}(t) + [G_{0,1}^{(0)}(t)\mathbf{x}],$$

  where "$[G_{0,1}^{(0)}\mathbf{x}]$" denotes the $(m \times r)$ array valued right sided dot product of the $(m \times r \times m)$ array $G_{0,1}^{(0)}$ and m-vector $\mathbf{x}$;

- Linear State-Dependent Poisson Amplitude Coefficient ($m \times q_0$):

$$H_0^{(0)}(\mathbf{x},t) = H_{0,0}^{(0)}(t) + [H_{0,1}^{(0)}(t)\mathbf{x}],$$

where "$[H_{0,1}^{(0)}\mathbf{x}]$" denotes the $(m \times q_0)$ array valued right sided dot product of the $(m \times q_0 \times m)$ array $H_{0,1}^{(0)}$ and m-vector $\mathbf{x}$;

- State-Independent Poisson Amplitude Coefficient ($m \times q_1 \times n$):

$$H_1^{(0)}(\mathbf{x},t) = H_{1,0}^{(0)}(t);$$

- Quadratic Control-Independent Cost Coefficient ((1×1), i.e., scalar):

$$C_0^{(0)}(\mathbf{x},t) = C_{0,0}^{(0)}(t) + \mathbf{C}_{0,1}^{(0)\top}(t)\mathbf{x} + \frac{1}{2}C_{0,2}^{(0)}(t) : \mathbf{x}\mathbf{x}^\top,$$

where the ":" denotes here the trace related, scalar valued, double dot product:

$$C_{0,2}^{(0)} : \mathbf{x}\mathbf{x}^\top = \text{Trace}[C_{0,2}^{(0)}\mathbf{x}\mathbf{x}^\top] = \mathbf{x}^\top C_{0,2}^{(0)}\mathbf{x},$$

for the $(m \times m)$ array $C_{0,2}^{(0)}(t)$;

- State-Linear Approximation of Control-Linear Cost Coefficient ($n \times 1$):

$$\mathbf{C}_1^{(0)}(\mathbf{x},t) = \mathbf{C}_{1,0}^{(0)}(t) + C_{1,1}^{(0)}(t)\mathbf{x};$$

- State-independent Approximation of Control-Quadratic Cost Coefficient ($n \times n$):

$$C_2^{(0)}(\mathbf{x},t) = C_{2,0}^{(0)}(t).$$

The machinery necessary to derive LSA above coefficients is from [13] where extensions of linear algebra for multilinear or even nonlinear algebra were developed in order to compensate for limitations in the standard linear algebra when encountering cubic and quartic array forms. The LSA coefficients are given by the following formulas (see [13] for details):

- **State Independent Coefficients**

$$\begin{aligned} F_{1,0}^{(0)}(t) = \overline{F_1}(t) &\equiv \frac{1}{|\mathcal{D}_\mathbf{x}|}\int_{\mathcal{D}_\mathbf{x}} F_1(\mathbf{x},t)d\mathbf{x}, \\ H_{1,0}^{(0)}(t) = \overline{H_1}(t) &\equiv \frac{1}{|\mathcal{D}_\mathbf{x}|}\int_{\mathcal{D}_\mathbf{x}} H_1(\mathbf{x},t)d\mathbf{x}, \\ C_{2,0}^{(0)}(t) = \overline{C_2}(t) &\equiv \frac{1}{|\mathcal{D}_\mathbf{x}|}\int_{\mathcal{D}_\mathbf{x}} C_2(\mathbf{x},t)d\mathbf{x}, \end{aligned} \tag{21}$$

where the size of the x-domain is $|\mathcal{D}_\mathbf{x}| \equiv \int_{\mathcal{D}_\mathbf{x}} d\mathbf{x}$, provided the x-domain, $\mathcal{D}_\mathbf{x}$, is bounded. If $\mathcal{D}_\mathbf{x}$ is discrete or has discrete components then we assume that the integrals are replaced by sums, $|\mathcal{D}_\mathbf{x}| = \sum_{\mathbf{j}_x} 1$ with $\mathbf{j}_x = [j_i]_{m\times 1}$.

- **Linear State Coefficients**
The linear approximations for the state-independent and linear coefficients that satisfy the following sets of equations:

$$\begin{aligned} F_{0,1}^{(0)}(t)\overline{\delta\mathbf{x}\delta\mathbf{x}^\top} &= \overline{\delta\mathbf{F}_0\delta\mathbf{x}}, & \mathbf{F}_{0,0}^{(0)}(t) &= \overline{\mathbf{F}_0}(t) - F_{0,1}^{(0)}(t)\overline{\mathbf{x}}; \\ H_{0,1}^{(0)}(t)\overline{\delta\mathbf{x}\delta\mathbf{x}^\top} &= \overline{\delta H_0\delta\mathbf{x}^\perp}, & H_{0,0}^{(0)}(t) &= \overline{H_0}(t) - [H_{0,1}^{(0)}(t)\overline{\mathbf{x}}]; \\ G_{0,1}^{(0)}(t)\overline{\delta\mathbf{x}\delta\mathbf{x}^\top} &= \overline{\delta G_0\delta\mathbf{x}^\perp}, & G_{0,0}^{(0)}(t) &= \overline{G_0}(t) - [G_{0,1}^{(0)}(t)\overline{\mathbf{x}}]; \\ C_{1,1}^{(0)}(t)\overline{\delta\mathbf{x}\delta\mathbf{x}^\top} &= \overline{\delta\mathbf{C}_1\delta\mathbf{x}^\perp}, & \mathbf{C}_{1,0}^{(0)}(t) &= \overline{\mathbf{C}_1}(t) - C_{1,1}^{(0)}(t)\overline{\mathbf{x}}; \end{aligned} \tag{22}$$

where the deviation from the mean value is defined by $\delta F \equiv F - \overline{F}$, for some function F. The "perp-posed" (symbol $\perp$) vector $\mathbf{x}^{\perp} = [x_k]_{1\times 1\times m}$ of the third array dimension changes column vectors into "plane" vectors of the third array dimension. The $\mathbf{x}^{\perp}$ appears with the $(m\times q)$ array H_0 in a covariance to properly balance the $(m \times q \times m)$ cubic array form $H^{(0)}_{0,1}(t)\overline{\delta\mathbf{x}\delta\mathbf{x}^{\top}}$, unlike the standard transpose. The equations for $G^{(0)}_{0,1}(t)$ and $C^{(0)}_{1,1}(t)$ have similar forms.

- **Quadratic State Coefficients**
  The system of equations for the approximate quadratic cost of control coefficients is determined by:

$$\begin{aligned} C^{(0)}_{0,0}(t) = \overline{C_0}(t) - \textstyle\sum_k \overline{x_k} C^{(0)}_{0,1,k}(t) - \frac{1}{2}\sum_k\sum_l \overline{x_k x_l} C^{(0)}_{0,2,k,l}(t), \\ \textstyle\sum_k \overline{\delta\mathbf{x}\delta x_k} C^{(0)}_{0,1,k}(t) + \frac{1}{2}\sum_k\sum_l \overline{\delta\mathbf{x}\delta(x_k x_l)} C^{(0)}_{0,2,k,l}(t), = \overline{C_0\delta\mathbf{x}}, \\ \textstyle\sum_k \overline{\delta(\mathbf{x}\mathbf{x}^{\top})\delta x_k} C^{(0)}_{0,1,k}(t) + \frac{1}{2}\sum_k\sum_l \overline{\delta(\mathbf{x}\mathbf{x}^{\top})\delta(x_k x_l)} C^{(0)}_{0,2,k,l}(t), = \overline{C_0\delta(\mathbf{x}\mathbf{x}^{\top})}. \end{aligned} \tag{23}$$

  Pivoted Gaussian elimination with high precision can be used to find a solution to the above equations.

3.2. Least Squares Approximation in HJB Equation

The least squares approximation (LSA) for the LQGP/U, the original coefficients in equations (1,12) are replaced by the above LSA approximate coefficients that generate a genuine LQGP problem. Hence the LSA satisfies the LQGP quadratic decomposition for the optimal value with respect to the state vector:

$$v^{(0)}(\mathbf{x},t) = E(t) + \mathbf{D}^{\top}(t)\mathbf{x} + \frac{1}{2}S(t) : \mathbf{x}\mathbf{x}^{\top}, \quad \mathbf{x} \in \mathcal{D}_{\mathbf{x}}, \tag{24}$$

which assumes use of use of the constrained optimal control vector, $\mathbf{u}^{(0)}(\mathbf{x},t)$, which is the argument of the minimum of (27). The unknown LQGP coefficient set includes the scalar $E(t)$, the m-vector $\mathbf{D}(t)$ and symmetric $(m\times m)$ array $S(t)$. This coefficient set needs only to be computed *once* for all $\mathbf{x}$ in the $\mathcal{D}_{\mathbf{x}}$ state domain *if a regular or constrained control is assumed*, since only integration in time is required, modulo the state-space floating point operations implied by equation (24). In fact, the time integration need only be for a single time step when model (24) is coordinated with another time stepping procedure. The final coefficient conditions, using equation (14) and comparing the same orders of $\mathbf{x}$, are that

$$E(t_f) = 0, \quad \mathbf{D}(t_f) = \mathbf{0}_{m\times 1}, \quad S(t_f) = S_f. \tag{25}$$

The LQGP optimal approximation, $v^{(0)}(\mathbf{x},t)$, satisfies the HJB equation:

$$\begin{aligned} 0 &= \dot{E}(t) + \dot{\mathbf{D}}^{\top}(t)\mathbf{x} + \frac{1}{2}\dot{S}(t) : \mathbf{x}\mathbf{x}^{\top} + \mathcal{S}^{(0)}(\mathbf{x},t) + C^{(0)}_{0,0} + \mathbf{C}^{(0)\top}_{0,1}\mathbf{x} + \frac{1}{2}C^{(0)}_{0,2} : \mathbf{x}\mathbf{x}^{\top} \\ &+ \left(\mathbf{F}^{(0)}_{0,0} + F^{(0)\top}_{0,1}\mathbf{x}\right)^{\top}(\mathbf{D} + S\mathbf{x}) + \frac{1}{2}(G^{(0)}_{0,0} + G^{(0)}_{0,1}\cdot\mathbf{x})(G^{(0)}_{0,0} + G^{(0)}_{0,1}\cdot\mathbf{x})^{\top} : S \\ &+ \sum_{l=1}^{q_0}\lambda_{0,l}\Big[\mathbf{D}^{\top}(\mathbf{H}^{(0)}_{0,0,l} + H^{(0)}_{0,1,l}\mathbf{x})\overline{Z_{0,l}} + \frac{1}{2}S : \Big(\Big((\mathbf{H}^{(0)}_{0,0,l} + H^{(0)}_{0,1,l}\mathbf{x})\mathbf{x}^{\top} \\ &+ \mathbf{x}(\mathbf{H}^{(0)}_{0,0,l} + H^{(0)}_{0,1,l}\mathbf{x})^{\top}\Big)\overline{Z_{0,l}} + (\mathbf{H}^{(0)}_{0,0,l} + H^{(0)}_{0,1,l}\mathbf{x})(\mathbf{H}^{(0)}_{0,0,l} + H^{(0)}_{0,1,l}\mathbf{x})^{\top}\overline{Z^2_{0,l}}\Big)\Big], \end{aligned} \tag{26}$$

where the control minimization terms have been collected in the control switching term,

$$\begin{aligned} \mathcal{S}^{(0)}(\mathbf{x},t) \equiv \operatorname*{Min}_{\mathbf{u}\in\mathcal{D}_{\mathbf{u}}}\Bigg[&(\mathbf{C}^{(0)}_{1,0} + C^{(0)}_{1,1}\mathbf{x})^{\top}\mathbf{u} + (F^{(0)}_{1,0}\mathbf{u})^{\top}(\mathbf{D} + S\mathbf{x}) + \frac{1}{2}\mathbf{u}^{\top}C^{(0)}_{2,0}\mathbf{u} \\ &+ \sum_{l=1}^{q_1}\lambda_{1,l}\Big[\mathbf{D}^{\top}[H^{(0)}_{1,0}\mathbf{u}]_l\overline{Z_{1,l}} + \frac{1}{2}S : \Big(\Big([H^{(0)}_{1,0}\mathbf{u}]_l\mathbf{x}^{\top} + \mathbf{x}[H^{(0)}_{1,0}\mathbf{u}]^{\top}_l\Big)\overline{Z_{1,l}} + [H^{(0)}_{1,0}\mathbf{u}]_l[H^{(0)}_{1,0}\mathbf{u}]^{\top}_l\overline{Z^2_{1,l}}\Big)\Big]\Bigg], \end{aligned} \tag{27}$$

where $\overline{Z_{i,l}} = \overline{Z_{i,l}}(t) \equiv \int_{\mathcal{Z}_{i,l}} z_l \phi_{i,l}(z_l,t) dz_l$ and $\overline{Z_{i,l}^2} = \overline{Z_{i,l}}(t) \equiv \int_{\mathcal{Z}_{i,l}} z_l^2 \phi_{i,l}(z_l,t) dz_l$ for $l = 1$ to q_i where $i = 0$ and 1. The jump vector coefficients are given by:

$$(\mathbf{H}^{(0)}_{0,0,l} + H^{(0)}_{0,1,l}\mathbf{x}) = (\mathbf{H}^{(0)}_{0,0,l}(t) + H^{(0)}_{0,1,l}(t)\mathbf{x}) = \mathbf{H}^{(0)}_{0,l}(\mathbf{x},t) \equiv [H^{(0)}_{0,i,l}(\mathbf{x},t)]_{m\times 1}, \tag{28}$$

for each $l = 1$ to q_0, and

$$[H^{(0)}_{1,0}(t)\mathbf{u}]_l \equiv [[H^{(0)}_{1,0}(t)\mathbf{u}]_{i,l}]_{m\times 1}, \tag{29}$$

for each $l = 1$ to q_1.

Note, that a regular or unconstrained control can not be found analytically due to the jump integral in (18) which shows the functional dependence of the control, $\mathbf{u}$, and optimal expected cost, $v^{(0)}(\mathbf{x},t)$. However, in the LQGP-LSA approximation a regular control, $\mathbf{u}^{(0)}_{\rm reg}$, can be determined since the form for $v^{(0)}(\mathbf{x},t)$ is known. The regular control is given by:

$$\mathbf{u}^{(0)}_{\rm reg} = \Big(C^{(0)}_{2,0} + \sum_{l=1}^{q_1} \lambda_{1,l} \overline{Z^2_{1,l}} H^{(0)\top}_{1,0,l} S H^{(0)}_{1,0,l}\Big)^{-1} \Big(\mathbf{C}^{(0)}_{1,0} + C^{(0)}_{1,1}\mathbf{x} + (F^{(0)\top}_{1,0} + \sum_{l=1}^{q_1} \lambda_{1,l}\overline{Z_{1,l}} H^{(0)\top}_{1,0,l})(\mathbf{D} + S\mathbf{x})\Big), \tag{30}$$

where $H^{(0)}_{1,0,l} \equiv [[H^{(0)}_{1,0}]_{i,l,j}]_{m\times n}$, so that, $H^{(0)\top}_{1,0,l} \equiv [[H^{(0)}_{1,0}]_{j,l,i}]_{n\times m}$. Note that the regular control is a linear function of $\mathbf{x}$.

The use of the LQGP-LSA approximation separates the dependence of state from the optimal expected cost, the so that only a system of ordinary differential equations in time needs to be solved provided that the switching term (constrained control) is known. For nonlinear control constraints, $\mathbf{u} \in \mathcal{D}_{\mathbf{u}}$, no closed form expression can be determined for the optimal control, $\mathbf{u}^{(0)}(\mathbf{x},t)$, in regions in which the regular control (30) is not valid. For constrained control, the LQGP quadratic decomposition only holds piecewise in separate subdomains of the state-space $\mathcal{D}_{\mathbf{x}}$, therefore the coefficient set must be calculated in each of the subdomains. The optimal control, $\mathbf{u}^*(\mathbf{x},t)$ is approximated for the pth subdomain $\mathcal{D}_{x_i}$ by

$$\mathbf{u}^*(\mathbf{x},t) \simeq \mathbf{u}^{(0)}_p(\mathbf{x},t) = \mathbf{u}^{(0)}_{p,0}(t) + u^{(0)}_{p,1}(t)\mathbf{x}. \tag{31}$$

In the case of linear constraints, (31) is an equality, as seen in the regular control (30). The coefficients for the approximation satisfy a set of equations similar to those for the *Linear State Coefficients* of the LSA (22) except that domain of $\mathbf{x}$ is now $\mathcal{D}_{x_i}$, and is given by:

$$u^{(0)}_{p,1}(t)\overline{\delta\mathbf{x}\delta\mathbf{x}^\top} = \overline{\delta\mathbf{u}^{(0)}_p \delta\mathbf{x}^\top}, \qquad \mathbf{u}^{(0)}_{p,0}(t) = \overline{\mathbf{u}^{(0)}_p}(t) - u^{(0)}_{p,1}(t)\overline{\mathbf{x}}. \tag{32}$$

As a result, the optimal control is valid in piecewise subdomains and continuous in the entire domain.

Upon substitution of the quadratic decomposition, (24), and the switching term (27) using the approximation (31) for the optimal control yields the LQGP HJB equation for subdomain $\mathcal{D}_{x_i}$:

$$\begin{aligned}
0 \;=\;& \dot{E}(t) + \dot{\mathbf{D}}^\top(t)\mathbf{x} + \frac{1}{2}\dot{S}(t) : \mathbf{x}\mathbf{x}^\top + C^{(0)}_{0,0} + \mathbf{C}^{(0)\top}_{0,1}\mathbf{x} + \frac{1}{2}C^{(0)}_{0,2} : \mathbf{x}\mathbf{x}^\top + (\mathbf{C}^{(0)}_{1,0} + C^{(0)}_{1,1}\mathbf{x})^\top(\mathbf{u}^{(0)}_{p,0} + u^{(0)}_{p,1}\mathbf{x}) \\
+\;& \frac{1}{2}(\mathbf{u}^{(0)}_{p,0} + u^{(0)}_{p,1}\mathbf{x})^\top C^{(0)}_{2,0}(\mathbf{u}^{(0)}_{p,0} + u^{(0)}_{p,1}\mathbf{x}) + \Big(\mathbf{F}^{(0)}_{0,0} + F^{(0)}_{1,0}\mathbf{u}^{(0)}_{p,0} + (F^{(0)\top}_{0,1} + F^{(0)}_{1,0}u^{(0)}_{p,1})\mathbf{x}\Big)^\top (\mathbf{D} + S\mathbf{x}) \\
+\;& \frac{1}{2}(G^{(0)}_{0,0} + G^{(0)}_{0,1}\cdot\mathbf{x})(G^{(0)}_{0,0} + G^{(0)}_{0,1}\cdot\mathbf{x})^\top : S \\
+\;& \sum_{l=1}^{q_0} \lambda_{0,l}\Big[\mathbf{D}^\top(\mathbf{H}^{(0)}_{0,0,l} + H^{(0)}_{0,1,l}\mathbf{x})\overline{Z_{0,l}} + \frac{1}{2}S : \Big(\Big((\mathbf{H}^{(0)}_{0,0,l} + H^{(0)}_{0,1,l}\mathbf{x})\mathbf{x}^\top \;+\; \mathbf{x}(\mathbf{H}^{(0)}_{0,0,l} + H^{(0)}_{0,1,l}\mathbf{x})^\top\Big)\overline{Z_{0,l}} \\
+\;& (\mathbf{H}^{(0)}_{0,0,l} + H^{(0)}_{0,1,l}\mathbf{x})(\mathbf{H}^{(0)}_{0,0,l} + H^{(0)}_{0,1,l}\mathbf{x})^\top\overline{Z^2_{0,l}}\Big)\Big] + \sum_{l=1}^{q_1}\lambda_{1,l}\Big[\mathbf{D}^\top[H^{(0)}_{1,0}(\mathbf{u}^{(0)}_{p,0} + u^{(0)}_{p,1}\mathbf{x})]_l\overline{Z_{1,l}} \\
+\;& \frac{1}{2}S : \Big(\Big([H^{(0)}_{1,0}(\mathbf{u}^{(0)}_{p,0} + u^{(0)}_{p,1}\mathbf{x})]_l\mathbf{x}^\top \;+\; \mathbf{x}[H^{(0)}_{1,0}(\mathbf{u}^{(0)}_{p,0} + u^{(0)}_{p,1}\mathbf{x})]^\top_l\Big)\overline{Z_{1,l}} \\
+\;& [H^{(0)}_{1,0}(\mathbf{u}^{(0)}_{p,0} + u^{(0)}_{p,1}\mathbf{x})]_l[H^{(0)}_{1,0}(\mathbf{u}^{(0)}_{p,0} + u^{(0)}_{p,1}\mathbf{x})]^\top_l\overline{Z^2_{1,l}}\Big)\Big].
\end{aligned} \tag{33}$$

Setting the state dependent terms, the zeroth order (ord(1)) terms, to zero, yields the following first order, coupled, array equation:

$$\begin{aligned} 0 &= \dot{E}(t) + C^{(0)}_{0,0} + \mathbf{C}^{(0)\top}_{1,0}\mathbf{u}^{(0)}_{p,0} + \frac{1}{2}\mathbf{u}^{(0)\top}_{p,0}C^{(0)}_{2,0}\mathbf{u}^{(0)}_{p,0} + (\mathbf{F}^{(0)}_{0,0} + F^{(0)}_{1,0}\mathbf{u}^{(0)}_{p,0})^\top\mathbf{D} \\ &+ \frac{1}{2}G^{(0)}_{0,0}G^{(0)\top}_{0,0} : S + \sum_{l=1}^{q_0}\lambda_{0,l}\left[\mathbf{D}^\top\mathbf{H}_{0,0,l}\overline{Z_{0,l}} + \frac{1}{2}S : \mathbf{H}_{0,0,l}\mathbf{H}^\top_{0,0,l}\overline{Z^2_{0,l}}\right] \\ &+ \sum_{l=1}^{q_1}\lambda_{1,l}\Big[\mathbf{D}^\top[H^{(0)}_{1,0}\mathbf{u}^{(0)}_{p,0}]_l\overline{Z_{1,l}} + \frac{1}{2}S : [H^{(0)}_{1,0}\mathbf{u}^{(0)}_{p,0}]_l[H^{(0)}_{1,0}\mathbf{u}^{(0)}_{p,0}]^\top_l\overline{Z^2_{1,l}}\Big). \end{aligned} \tag{34}$$

Upon equating like orders of powers of the state $\mathbf{x}$, first order (ord($\mathbf{x}^\top$)) terms satisfy the following equation:

$$\begin{aligned} 0 &= \dot{\mathbf{D}}(t) + \mathbf{C}^{(0)}_{0,1} + u^{(0)\top}_{p,1}\mathbf{C}^{(0)}_{1,0} + C^{(0)\top}_{1,1}\mathbf{u}^{(0)}_{p,0} + u^{(0)\top}_{p,1}C^{(0)}_{2,0}\mathbf{u}^{(0)}_{p,0} + S(\mathbf{F}^{(0)}_{0,0} + F^{(0)}_{1,0}\mathbf{u}^{(0)}_{p,0}) \\ &+ (F^{(0)\top}_{0,1} + F^{(0)}_{1,0}u^{(0)}_{p,1})\mathbf{D} + G^{(0)\perp}_{0,1} : SG^{(0)}_{0,0} + \sum_{l=1}^{q_0}\lambda_{0,l}\left[\left(H^\top_{0,1,l}\mathbf{D} + S\mathbf{H}_{0,0,l}\right)\overline{Z_{0,l}} + H^\top_{0,1,l}S\mathbf{H}_{0,0,l}\overline{Z^2_{0,l}}\right] \\ &+ \sum_{l=1}^{q_1}\lambda_{1,l}\Big[[H^{(0)}_{1,0}u^{(0)}_{p,1}]^\top_l\mathbf{D}\overline{Z_{1,l}} + S[H^{(0)}_{1,0}\mathbf{u}^{(0)}_{p,0}]_l\overline{Z_{1,l}} + [H^{(0)}_{1,0}u^{(0)}_{p,1}]_lS[H^{(0)}_{1,0}\mathbf{u}^{(0)}_{p,0}]_l\overline{Z^2_{1,l}}\Big], \end{aligned} \tag{35}$$

where

$$[H^{(0)}_{1,0}u^{(0)}_{p,1}]_l \equiv [[H^{(0)}_{1,0}u^{(0)}_{p,1}]_{i,l,j}]_{m\times m}, \tag{36}$$

for each $l = 1$ to q_1. The x-gradient, $\nabla_x[\bullet]$, of the right hand side of LQGP HJB equation (33) was used, followed by setting remaining $\mathbf{x}$ terms to zero.

Finally, the second order (ord($\frac{1}{2}\mathbf{x}\mathbf{x}^\top$)) satisfies the following array equation:

$$\begin{aligned} 0 &= \dot{S}(t) + C^{(0)}_{0,2} + C^{(0)\top}_{1,1}u^{(0)}_{p,1} + u^{(0)\top}_{p,1}C^{(0)}_{1,1} + u^{(0)\top}_{p,1}C^{(0)}_{2,0}u^{(0)}_{p,1} + (F^{(0)\top}_{0,1} + F^{(0)}_{1,0}u^{(0)}_{p,1})^\top S \\ &+ S(F^{(0)\top}_{0,1} + F^{(0)}_{1,0}u^{(0)}_{p,1}) + G^{(0)\perp}_{0,1} : SG^{(0)}_{0,1} + \sum_{l=1}^{q_0}\lambda_{0,l}\left[\left(SH_{0,1,l} + H^\top_{0,1,l}S\right)\overline{Z_{0,l}} + H^\top_{0,1,l}SH_{0,1,l}\overline{Z^2_{0,l}}\right] \\ &+ \sum_{l=1}^{q_1}\lambda_{1,l}\Big[(S[H^{(0)}_{1,0}u^{(0)}_{p,1}]_l[H^{(0)}_{1,0}u^{(0)}_{p,1}]^\top_lS)\overline{Z_{1,l}} + [H^{(0)}_{1,0}u^{(0)}_{p,1}]^\top_lS[H^{(0)}_{1,0}u^{(0)}_{p,1}]_l\overline{Z^2_{1,l}}\Big)\Big], \end{aligned} \tag{37}$$

which can be calculated by taking the $\nabla_x[\nabla^\top_x[\bullet]]$ operation of the right hand side of the LQGP HJB equation. Note that the "perp" ($\perp$) array operation is used to put the three subscript, first order diffusion or Gaussian noise coefficient, $G^{(0)}_{0,1}$, in the proper form for the array dimensions of the $S(t)$ and $\mathbf{D}(t)$ equations.

The coefficients in the equations (34,35,37) need to be discretized according to the finite difference scheme in (42). Note, that this set of three coupled ordinary differential equations only needs to be solved once for each subdomain $\mathcal{D}_{x_i}$ for a given time t, and assembled over the entire domain $\mathcal{D}_{\mathbf{x}}$ to form $v^{(0)}(\mathbf{x}, t)$. This provides additional information for all predictor-corrector iterations that should help to accelerate convergence. For the situation that considers the unconstrained or regular control, the resulting system of equations is nonlinear (see [11]).

3.3. Gauss-Statistics Quadrature for Jump Integrals

A derivation of the quadrature method, called *Gauss-Statistics Quadrature*, is in Westman and Hanson [13]. Here, we will present the necessary material for the method for the sake of completeness. A numerical quadrature for the arbitrary jump densities $\phi_{i,l}(z_l, T_k)$ in the HJB equation (17), $i = 0$, and the control switching term (18), $i = 1$, with sufficiently high accuracy is needed. For simplicity, the jump subscript and time dependence is omitted, seeking an approximation of the mean jump defining integral:

$$\overline{f(Z)} \equiv \int_{\mathcal{Z}} f(z)\phi(z)dz \simeq \sum_l w_l f(z_l), \tag{38}$$

for some integrable function $f(z)$ with respect to the density $\phi(z)$, where $z_l \in \mathcal{Z}$ are the nodes and w_l are the corresponding weights.

A two-point Gauss-Statistics quadrature is necessary, since the one-point version does not possess sufficient accuracy. The two-point rule has the form:

$$\overline{f(Z)} \equiv \int_{\mathcal{Z}} f(z)\phi(z)dz \simeq w_1 f(z_1) + w_2 f(z_2), \quad \text{for} \quad z_1 < z_2 \in \mathcal{Z}, \tag{39}$$

and possesses third moment accuracy such that,

$$\begin{aligned} 1 &= w_1 + w_2, \\ \overline{Z} &= w_1 z_1 + w_2 z_2, \\ \overline{Z^2} &= w_1 z_1^2 + w_2 z_2^2 = (\overline{Z})^2 + \mathrm{Var}[Z], \\ \overline{Z^3} &= w_1 z_1^3 + w_2 z_2^3 = (\overline{Z})^3 + 3\overline{Z}\mathrm{Var}[Z] + \overline{(\delta Z)^3}, \end{aligned} \tag{40}$$

which consists of four equations in the four unknown quadrature parameters $\{w_1, w_2, z_1, z_2\}$. In equation (40), the basic moments have been related to the relevant central moments in terms of the deviation $\delta Z \equiv Z - \overline{Z}$, so that $\overline{\delta Z} = 0$, $\mathrm{Var}[Z] \equiv \overline{(\delta Z)^2}$, to systematize the algebraic simplifications. The quadrature parameter formulas are found to be:

$$\begin{aligned} z_1 &= \overline{Z} + \left(\gamma/2 - \sqrt{(\gamma/2)^2 + 1}\right)\sigma, \quad & z_2 &= \overline{Z} + \left(\gamma/2 + \sqrt{(\gamma/2)^2 + 1}\right)\sigma, \\ w_1 &= \tfrac{1}{2}\left(1 + \gamma/\left(2\sqrt{(\gamma/2)^2 + 1}\right)\right), \quad & w_2 &= \tfrac{1}{2}\left(1 - \gamma/\left(2\sqrt{(\gamma/2)^2 + 1}\right)\right), \end{aligned} \tag{41}$$

using the notation that $\sigma \equiv \sqrt{\mathrm{Var}[Z]}$ is the standard deviation and the normalized skewness of the density is $\gamma \equiv \mathrm{Skew}[Z] \equiv \overline{(\delta Z)^3}/(\mathrm{Var}[Z])^{\frac{3}{2}}$ (see [14] for a general definition of skewness).

3.4. Finite Difference Discretization Scheme

A second order central finite difference discretization scheme in the state and backward time is implemented using a compact subscript notation for (17,18) is given by:

$$\begin{gathered} \mathbf{x} \to \mathbf{X_j} = [X_{i,j_i}]_{m\times 1} = [X_{i0} + j_i \cdot DX_i]_{m\times 1}, \\ \mathbf{j} = [j_i]_{m\times 1}, \text{ where } j_i = 0 \text{ to } M_i, \text{ for } i = 1 \text{ to } m; \\ t \to T_k = t_f - k * DT \text{ for } k = 0 \text{ to } K; \\ v(\mathbf{X_j}, T_k) \to V_{\mathbf{j}k}; \\ \tfrac{\partial v}{\partial t}(\mathbf{X_j}, T_k) \to (V_{\mathbf{j},k+\frac{1}{2}+\frac{1}{2}} - V_{\mathbf{j},k+\frac{1}{2}-\frac{1}{2}})/(-2 \cdot \tfrac{1}{2}DT); \\ \nabla_x v(\mathbf{X_j}, T_k) \to \mathbf{DV}_{\mathbf{j}k}; \quad \nabla_x \nabla_x^\top v(\mathbf{X_j}, T_k) \to DDV_{\mathbf{j}k}; \\ [v(\mathbf{X_j} + [H_{0,i,j}(\mathbf{X_j}, T_k)z_j]_{m\times 1}, T_k)]_{q_0\times 1} \to \mathbf{HV0}_{\mathbf{j}k} = [HV0_{i,\mathbf{j}k}]_{q_0\times 1}; \\ [v(\mathbf{X_j} + [[H_1(\mathbf{X_j}, T_k)\mathbf{u}]_{i,j}z_j]_{m\times 1}, T_k)]_{q_1\times 1} \to \mathbf{HV1}_{\mathbf{j}k} = [HV1_{i,\mathbf{j}k}]_{q_1\times 1}; \\ F_\ell(\mathbf{X_j}, T_k) \to F\{\ell\}_{\mathbf{j}k} \text{ for } \ell = 0, 1; \quad C_\ell(\mathbf{X_j}, T_k) \to C\{\ell\}_{\mathbf{j}k} \text{ for } \ell = 0, 1, 2; \\ \begin{aligned} \mathcal{S}^*(\mathbf{X_j}, T_k) \to \mathcal{S}_{\mathbf{j}k} \quad &= \quad \min_{\mathbf{u}\in\mathcal{D}_\mathbf{u}} \Big[(F1_{\mathbf{j}k}\mathbf{u})^T\mathbf{DV}_{\mathbf{j}k}; + \mathbf{C1}_{\mathbf{j}k}^T\mathbf{u}(t) + \tfrac{1}{2}\mathbf{u}^T(t)C2_{\mathbf{j}k}\mathbf{u}(t) \\ &+ \quad \textstyle\sum_{i=1}^{q_1} \lambda_{1,i} \int_{\mathcal{Z}_{1i}} [HV1_{i,\mathbf{j}k} - V_{\mathbf{j}k}]\, \phi_{1,i}(z_i, T_k)dz_i \Big]. \end{aligned} \end{gathered} \tag{42}$$

In the above scheme, second order central differencing is employed to approximate the derivatives of $v(\mathbf{x}, t)$ with respect to $\mathbf{x}$ at interior points, with one-sided differences of the same order for boundary points. The

delay terms associated with the Poisson noise processes, $\mathbf{HV0}_{\mathbf{j}k}$ and $\mathbf{HV1}_{\mathbf{j}k}$, are computed using linear interpolation that has the same order of accuracy as that used for the derivatives.

In the case of linear dynamics, quadratic costs and Gaussian noise (the LQG problem, see [1, 2, 8]) formal solutions exist that require numerical solution for a state-independent matrix Riccati equation. Further, if there are Poisson noise terms (the LQGP problem, see [15, 11]), then formal solutions exist that are quadratic state-space decompositions, that require the numerical solution for a system of genuine nonlinear differential equations in time. In order to obtain a numerical solution of the LQGP/U problem, a predictor–corrector method is used, since the HJB equation is quadratic in the gradient, $\nabla_x[v](\mathbf{x},t)$, resulting from quadratic costs. The predictor–corrector method is combined with a Crank-Nicolson method so that the local truncation error is second order in both the state and time derivatives, coupled with linear interpolation of equivalent accuracy for the functional Poisson jump integrals. At each time step, an initial approximation is used that should accelerate convergence. This approximation utilizes a least squares approximation (LSA) for a comparable linear system (LQGP in both state and control) under the same restrictions for the state and control. This Crank-Nicolson, predictor–corrector method is a modification for stochastic dynamic programming described by Hanson et al. [5, 6, 9]. This method with modifications is based on an earlier [3] Crank–Nicolson, predictor–corrector method for standard parabolic partial differential equations. The numerical treatment here advances that of [12]. Here we consider the more general case of constrained control providing the necessary changes in order to obtain a solution, in our previous efforts [11, 12, 13] we assumed regular control.

Combining the Gauss-Statistics quadrature for the Poisson jump integrals with approximation of the state-space dependence by linear interpolation of the Poisson jump advanced optimal value and using the discretization scheme (42) for the HJB equation (17) yields,

$$\sum_{l=0}^{q_0}\lambda_{0,l}\int_{\mathcal{Z}_{0,l}}[\mathbf{HV0}_{p,l,\mathbf{j},k-1/2}-V_{\mathbf{j},k-1/2}]\phi_{0,l}(z_l,T_{k-1/2})dz_l$$
$$\simeq\sum_{l=0}^{q_0}\lambda_{0,l}\left[\sum_{p=1}^{2}w_{p,l,k-1/2}\mathrm{LIV0}_{p,l,\mathbf{j},k-1/2}-V_{\mathbf{j},k-1/2}\right], \tag{43}$$

and for the control switching term (18),

$$\sum_{l=0}^{q_1}\lambda_{1,l}\int_{\mathcal{Z}_{1,l}}[\mathbf{HV1}_{p,l,\mathbf{j},k-1/2}-V_{\mathbf{j},k-1/2}]\phi_{1,l}(z_l,T_{k-1/2})dz_l$$
$$\simeq\sum_{l=0}^{q_1}\lambda_{1,l}\left[\sum_{p=1}^{2}w_{p,l,k-1/2}\mathrm{LIV1}_{p,l,\mathbf{j},k-1/2}-V_{\mathbf{j},k-1/2}\right], \tag{44}$$

where subscripts have been restored for consistency, e.g., $w_{p,l,k-1/2}$ exhibits the dependence of the Gauss-Statistics pth weight parameter on the full dependence of the density $\phi_{0,l}(z_l,T_{k-1/2})$, including the Poisson and time indices. Here, $\mathrm{LIV0}_{p,l,\mathbf{j},k-1/2}=\mathrm{HV00}_{p,l,\mathbf{j},k-1/2}$ denotes the linear interpolation for $v(\mathbf{x_j}+\mathbf{H}_{0,l}(\mathbf{x_j},T_{k-1/2})z_{p,l},T_{k-1/2})\simeq\mathrm{HV0}_{p,l,\mathbf{j},k-1/2}$ in terms of sufficiently close $V_{\mathbf{j}'_{p,l,\mathbf{j},k-1/2},k-1/2}$ optimal values in the neighborhood of $(\mathbf{x_j}+\mathbf{H}_{0,l}(\mathbf{x_j},T_{k-1/2})z_l)\simeq\mathbf{x}_{\mathbf{j}'_{p,l,\mathbf{j},k-1/2}}$, similarly for $LIV1$ and HV01.

3.5. Predictor-Corrector Iteration Scheme

In this section, we present the modifications to the iterative algorithm of Westman and Hanson [13] to accommodate general control constraints, as well as the bilinear term in the control and Poisson noise, $[H_1(\mathbf{X}(t),t)\mathbf{U}(t)]d\mathbf{P}_1(t)$, of the defining SDE (1). In [12], a regular control was assumed, and the method presented here will reduce to that algorithm if a regular control is assumed and the bilinear term in the control and Poisson noise is absent, ie. $H_1\equiv 0$.

It is assumed that the prior value for the optimal, expected performance index, $V_{\mathbf{j},k-1}$, and the LQGP coefficient set, $\{E_{k-1},D_{k-1},S_{k-1}\}$, is known prior to calculating the new time step k for $k=1$ to K. For each new time step, $k=1$ to K, the algorithm starts by determining the solution for the LQGP least squares approximation. This approximation is formed in a piecewise fashion since the optimal control, $\mathbf{u}^{(0)}(\mathbf{x},t)$, is

approximated in subdomains of the state-space $\mathcal{D}_{\mathbf{x}}$. Therefore, the LQGP least squares approximation for each subdomain $\mathcal{D}_{x_i}$ is determined in the following way:

1. Determine the approximation for $\mathbf{u}_{i,k}^{(0)}$ from (31) using (32) and (27) with D_{k-1} and S_{k-1}.
2. Calculate the values for E_k, D_k,and S_k from (34, 35,37) using $\overline{\mathbf{u}_{i,k}^{(0)}}$ subject to the final conditions E_{k-1}, D_{k-1},and S_{k-1}.
3. Assemble $V_{\mathbf{j},k}^{(\text{lqgp})} = v_{\mathbf{j},k}^{(0)}$ from (24) using E_k, D_k,and S_k.

As a requirement of the Crank–Nicolson procedure used for the second order approximate time derivative, a number of quantities must be determined at the mid-point of the time step, $T_{k-\frac{1}{2}}$. Next the convergence accelerating *extrapolator* (x) *start* is formed:

$$\begin{gathered} V_{\mathbf{j},k-1/2}^{(\text{x})} = \left\{ \begin{array}{ll} \frac{1}{2}(V_{\mathbf{j},k}^{(\text{lqgp})} + V_{\mathbf{j},k-1}), & k=1 \\ \frac{1}{8}(3V_{\mathbf{j},k}^{(\text{lqgp})} + 6V_{\mathbf{j},k-1} - V_{\mathbf{j},k-2}), & k \geq 2 \end{array} \right\}; \\ \text{Calculate new values}: \ \mathbf{DV}_{\mathbf{j},k-1/2}^{(\text{x})}; \ \ \text{DDV}_{\mathbf{j},k-1/2}^{(\text{x})}; \ \ \text{LIV0}_{\mathbf{j},k-1/2}^{(\text{x})}; \ \ \text{LIV1}_{\mathbf{j},k-1/2}^{(\text{x})}; \ \ \mathcal{S}_{\mathbf{j},k-1/2}^{(\text{x})}, \end{gathered} \tag{45}$$

with the final condition $V_{\mathbf{j},0} = v(\mathbf{x}_{\mathbf{j}}, t_f)$ from equation (12). The switching term, $\mathcal{S}_{\mathbf{j},k-1/2}^{(\text{x})}$, is determined by solving the optimization problem (18) which utilizes Gauss-Statistics quadrature (44) to evaluate the jump integral. In Hanson et al. [5, 6, 9], the first time step does not use an accelerating extrapolator start since only the values at t_f are known. The first step is the most crucial time step and usually has the most difficulty converging and takes the most iterations to converge. Using a LQGP least squares approximation in the extrapolator start (45) should aid in convergence for the solution of the HJB equation. After the first time step, $k \geq 2$, a quadratic extrapolation is used here that has cubic accuracy for evenly spaced state nodes.

The *corrector* (xpec) *step* determines the value of V at the new time time step T_k:

$$\begin{aligned} V_{\mathbf{j},k}^{(\text{xpec},\gamma)} &= V_{\mathbf{j},k-1} + DT \cdot \left[\left(C0 + F0^{\top}\mathbf{DV} + \tfrac{1}{2}(G_0 G_0^{\top}) : DDV \right)_{\mathbf{j},k-1/2}^{(\text{xpece},\gamma-1)} \right. \\ &+ \left. \mathcal{S}_{\mathbf{j},k-1/2}^{(\text{xpece},\gamma-1)} + \textstyle\sum_{l=1}^{q_0} \lambda_{0,l} \left[\sum_{p=1}^{2} w_{p,l,k-1/2} \text{LIV0}_{p,l,\mathbf{j},k-1/2}^{(\text{xpece},\gamma-1)} - V_{\mathbf{j},k-1/2}^{(\text{xpece},\gamma-1)} \right] \right], \end{aligned} \tag{46}$$

for $\gamma = 0$ to $\gamma_{\max}$ while the convergence criterion is still unsatisfied. The *extrapolated-predictor* (xp) *step* is the starting corrector step: $V_{\mathbf{j},k}^{(\text{xp})} = V_{\mathbf{j},k}^{(\text{xpec},0)}$, and the input is the extrapolation (x) start: $V_{\mathbf{j},k-1/2}^{(\text{xpece},-1)} \equiv V_{\mathbf{j},k-1/2}^{(\text{x})}$.

The *corrector evaluation* (xpece) *step* is used to determine the various quantities at the half time step, $T_{k-\frac{1}{2}}$, based on the corrector step $V_{\mathbf{j},k}^{(\text{xpec},\gamma)}$ and previous step value $V_{\mathbf{j},k-1}$ to form the Crank-Nicolson average (i.e., precision consistent linear interpolation):

$$\begin{gathered} V_{\mathbf{j},k-1/2}^{(\text{xpece},\gamma)} = \tfrac{1}{2}(V_{\mathbf{j},k-1} + V_{\mathbf{j},k}^{(\text{xpec},\gamma)}); \\ \text{Calculate new values}: \ \mathbf{DV}_{\mathbf{j},k-1/2}^{(\text{xpece},\gamma)}; \ \ \text{DDV}_{\mathbf{j},k-1/2}^{(\text{xpece},\gamma)}; \ \ \text{LIV0}_{\mathbf{j},k-1/2}^{(\text{xpece},\gamma)}; \ \ \text{LIV1}_{\mathbf{j},k-1/2}^{(\text{xpece},\gamma)}; \ \ \mathcal{S}_{\mathbf{j},k-1/2}^{(\text{xpece},\gamma)}. \end{gathered} \tag{47}$$

Upon convergence the final corrector step is used as the value for $v(\mathbf{X}_{\mathbf{j}}, T_k)$, i.e., set $V_{\mathbf{j},k} = V_{\mathbf{j},k}^{(\text{xpec},\gamma^*)}$, where $\gamma^* \leq \gamma_{\max}$ is the number of corrections when the corrections are stopped according to the stopping criterion. Finally the constrained stochastic optimal control, $\mathbf{u}^*$, for time T_k needs to be determined by solving for the argument of the minimum of (18) using $\mathbf{DV}_{\mathbf{j},k}$ and $\text{LIV1}_{\mathbf{j},k}$. Stopping criterion and mesh ratio selection can be found in [13].

4. Conclusions

These results are of great interest because the LQGP/U problem is a *nonlinear* theoretical model and a canonical computational model. This control model features general control constraints and Poisson noise that is linear in the control. The state space representation of the LSA for the nonlinear LQGP/U problem forms a genuine LQGP problem, which should accelerate the convergence of the iterative method used

to approximate the solution for the *LQGP/U HJB equation*. The Poisson components permit the model to account for a wide variety of large random fluctuations, whereas Gaussian noise is useful for relatively less severe background fluctuations. The stochastic differential equation formulation used here makes the construction of models analogous to that for other dynamic systems. The features of this control model allow for greater realism without having to compromise aspects of physical reality than does the corresponding linear model. This problem formulation would be useful for investigating the response of systems subject to extreme random inputs or catastrophic failure.

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