APPLICATIONS ORIENTED MATHEMATICS
PRELIMINARY EXAMINATION

Friday, April 27, 2001 1:00-4:00pm

Each of the 8 numbered questions is worth 20 points. All questions will be graded, but your score for the examination will be the sum of the scores of your best **FIVE** questions.

Use a separate answer booklet for each question, and do not put your name on the answer booklets, instead put the number that is on the envelope the exam was in. When you have completed the examination, insert all your answer booklets in the envelope provided. Then seal and print your name on the envelope.

1. Find the leading term in the asymptotic expansion of the following integral $I(\lambda)$ as (a) $\lambda \to \infty$ and (b) $\lambda \to -\infty$

$$I(\lambda) = \int_0^{\pi/2} e^{\lambda(\sin t - t)}(1 + e^t)dt.$$

2. Find the leading term in the asymptotic expansion of the following integral $I(\lambda)$ as $\lambda \to \infty$ if (a) $N = 0$, (b) $N = 1$ and (c) $N = 2$

$$I(\lambda) = \int_0^{\pi} e^{i\lambda t^2} t^N dt.$$

3. Consider the ODE $y''(x) = x^2 y(x)$. Find the leading term in the expansions of all solutions as $x \to \infty$ (show you work!).

4. Consider the eigenvalue problem

$$y''(x) + \frac{\lambda}{2}(1 - x^2)y(x) = 0 \quad , y(0) = 0, \quad y(\infty) = 0.$$

Find, using the WKB method, the leading term in the expansion of the large eigenvalues λ_n as $n \to \infty$. Hint: the solution of $Y''(\xi) = \xi Y(\xi), Y(\infty) = 0$ is $Y(\xi) = CAi(\xi)$, where Ai is the Airy function, which satisfies

$$Ai(\xi) \sim \frac{1}{2\sqrt{\pi}} \xi^{-1/4} \exp(-\frac{2}{3}\xi^{3/2}), \quad \xi \to \infty,$$

$$Ai(\xi) \sim \frac{1}{\sqrt{\pi}}(-\xi)^{-1/4} \sin(\frac{2}{3}(-\xi)^{3/2} + \frac{\pi}{4}), \quad \xi \to -\infty.$$

5. Consider the differential equation for $u(x)$:

$$\varepsilon u'' + xu = -x$$

$$u(-1) = 0, \quad u(1) = 0$$

Construct an asymptotic approximation to u if $\varepsilon \ll 1$.

6. Let $v(t)$ satisfy

$$v'' + v - \frac{1}{6}\varepsilon v^3 = 0, \quad 0 < \varepsilon \ll 1,$$

$$v(0) = 1, \quad v'(0) = 0$$

   (a) Show and explain the failure of a regular expansion for large times.

   (b) Construct an approximation to v that is valid for $t = \mathcal{O}(1/\varepsilon)$. Give a two term expansion for the period of v.

7. Consider the ODE

$$y'' + y' + \lambda y + y^3 = 0, \quad t > 0, \quad \lambda \text{ is real.}$$

Determine the bifurcation points, find the steady-state solutions (include a bifucation diagram), and determine the stability of each branch.

8. Consider the PDE

$$\varepsilon \Delta u + (y - x)u_x - (x + y)u_y = 0, \;\; x^2 + y^2 < 1$$

$$u(x, y) = f(\theta), \;\; x^2 + y^2 = 1$$

Use the method of matched asymptotic expansions to find the leading term in the composite expansion for $u(x, y)$ for $0 < \varepsilon \ll 1$. If your solution is not unique, explain briefly in words how a unique solution can be obtained.