APPLICATIONS ORIENTED MATHEMATICS PRELIMINARY EXAMINATION

Wednesday, April 23, 2003 1:00-4:00pm

Each of the 8 numbered questions is worth 20 points. All questions will be graded, but your score for the examination will be the sum of the scores of your best **FIVE** questions.

Use a separate answer booklet for each question, and do not put your name on the answer booklets, instead put the number that is on the envelope the exam was in. When you have completed the examination, insert all your answer booklets in the envelope provided. Then seal and print your name on the envelope.

1. Find the leading term in the asymptotic expansion of the following integrals as $x \to +\infty$

$$\text{(a)} \int_{-\infty}^{\infty} e^{-xt^2(t-1)^2} dt \qquad \text{(b)} \int_0^1 \frac{1}{(1+t^3)^x} dt.$$

2. Consider the following integral as $\lambda \to +\infty$

$$I(\lambda) = \int_C e^{\lambda(z - z^4/4)} dz.$$

Here C goes from $-i\infty$ to $i\infty$ in the complex z plane.

   (a) Find all saddle points in the complex z plane.

   (b) At each saddle, find the local steepest descent (SD) and steepest ascent (SA) directions.

   (c) Find the (global) constant phase contours through each saddle, and identify the SD and SA contours.

   (d) Which saddle point(s) determine the asymptotic behavior of the integral? (You do not have to actually compute the asymptotic expansion.)

3. Consider the ODE

$$y''(x) + \frac{2}{x} y'(x) - y(x) = 0.$$

Find the leading term in the expansions of all solutions as $x \to \infty$ (show your work!).

4. Consider the eigenvalue problem

$$y''(x) + \lambda(x+1)^4\, y(x) = 0,\ 0 < x < 1,\ \ y(0) = 0,\ y'(1) = 0.$$

Find, using the WKB method, the leading term in the expansion of the large eigenvalues λ_n as $n \to \infty$. Also obtain the corresponding eigenfunctions.

5. Use singular perturbation methods to construct a leading order composite or uniform approximation to $y(x)$ as $\varepsilon \to 0^+$

$$\varepsilon y'' + 4x^3 y' + \varepsilon(1+x)y = 0, \quad y(-1) = 1, \quad y(1) = 2.$$

6. Find the leading order approximation for $\varepsilon \to 0^+$

$$\varepsilon \Delta u - r u_r = 0, \ \text{ in } 1 < x^2 + y^2 < 9,$$

with the boundary conditions

$$u = A(\theta) \ \text{ on } x^2 + y^2 = 1; \quad u = B(\theta) \ \text{ on } x^2 + y^2 = 9.$$

7. Find the first 2 terms in the expansion of the smallest eigenvalue and its normalized eigenfunction, for the perturbed eigenvalue problem

$$y'' + (\lambda + \varepsilon x)y = 0, \quad y(0) = 0, y(1) = 0.$$

8. Find the leading term (valid for large t) in the asymptotic expansion of $x(t)$ in the case of small amplitude oscillations ($\varepsilon \ll 1$) of the problem

$$x'' + \sin x = 0, \quad x(0) = \varepsilon, \ \ x'(0) = 0.$$