APPLICATIONS ORIENTED MATHEMATICS
PRELIMINARY EXAMINATION

Wednesday, September 6, 2000 11:00-2:00pm

Each of the 8 numbered questions is worth 20 points. All questions will be graded, but your score for the examination will be the sum of the scores of your best **FIVE** questions.

Use a separate answer booklet for each question, and do not put your name on the answer booklets, instead put the number that is on the envelope the exam was in. When you have completed the examination, insert all your answer booklets in the envelope provided. Then seal and print your name on the envelope.

1. Consider

$$I(x) = \int_0^1 \frac{te^{xt}}{1 + e^{t^2}} dt$$

Find the leading term in the asymptotic expansion of $I(x)$ as (a) $x \to +\infty$, (b) $x \to -\infty$.

2. Consider

$$I(x) = \frac{1}{2\pi i} \int_C \frac{e^{x(t^2/2 - at)}}{t} dt$$

where a is real and C is a vertical contour that goes from $b - i\infty$ to $b + i\infty$ in the t-complex plane, where $b > 0$. Find the leading term as $x \to +\infty$ for (a) $a > 0$, (b) $a < 0$.

3. Consider the ODE

$$y''(x) = (x^4 + x)y(x).$$

Find the leading behavior of *all* solutions as $x \to +\infty$.

4. Consider the BVP

$$y''(x) + \lambda e^{2x} y(x) = 0, \quad 0 < x < 1,$$
$$y(0) = 0, \quad y'(1) = 0$$

Use the WKB method to find an approximation for the large eigenvalues (i.e. compute λ_n for n large). Also give an approximation to the eigenfunctions.

5. Consider

$$y'' + yy' - \lambda y + y^3 = 0$$

Describe the steady states, locate and classify the bifurcation points, determine the stability of each branch, and sketch the bifurcation diagram.

6. Consider

$$\begin{aligned} \frac{dx}{dt} &= x - xy, \quad x(0) = .5 \\ \varepsilon \frac{dy}{dt} &= x - y, \quad y(0) = 0 \end{aligned}$$

   (a) Find the leading term in the composite or uniform expansion as $\varepsilon \to 0^+$.

   (b) Make a rough sketch of the solution in the xy phase plane.

7. Compute a leading order composite or uniform approximation as $\varepsilon \to 0^+$:

$$4\varepsilon y'' + 6\sqrt{x}y' - 3y = -3; \quad y(0) = 0, \quad y(1) = 3.$$

8. Use the multiple scales method to obtain an asymptotic approximation ($\varepsilon \to 0^+$) to $y = y(t)$ that is valid for all times $t = \mathcal{O}(\varepsilon^{-1})$

$$y'' + \varepsilon \left[-y + \frac{1}{3}(y')^3 \right] + y = 0; \quad y(0) = 1; \quad y'(0) = 0.$$