MATHEMATICAL SCIENCE PRELIMINARY EXAMINATION

Friday, April 26, 2002 1:00-4:00pm

The exam is based on questions from the areas: **Fluid Dynamics** and **Computational Finance**. There are 4 questions in each area. Each question is worth 20 points. All questions will be graded, but your score for the examination will be the sum of the scores of your best **FIVE** questions.

Use a separate answer booklet for each question, and do not put your name on the answer booklets, instead put the number that is on the envelope the exam was in. When you have completed the examination, insert all your answer booklets in the envelope provided. Then seal and print your name on the envelope.

Computational Finance

1. The price of a non-dividend paying asset $S(t)$ is modelled by the SDE:

$$dS = \mu(S,t)dt + \sigma S dW$$

where $W(t)$ is Brownian motion, μ is the real world return, and σ is the volatility. Also, assume r is the risk-free interest rate.

   (a) *Derive* a Black-Scholes type equation for $c(S,t)$, the price of a European call option on this asset with expiration T and strike price E. Be sure to state the final pricing problem completely.

   (b) If you need to replicate the call option using the asset and bonds, give the amount of each that is needed at time t to be perfectly hedged.

   (c) If the asset pays a continuous yield of q, derive a new Black-Scholes equation that takes this into account.

2. Let X be normal with mean m and variance $\sigma^2 t$. Compute $\mathrm{E}[e^X]$. Note: This is a common calculation in computational finance.

3. The risk-neutral dynamics of a stock price $S(t)$ is given by

$$dS = rSdt + \sigma S dW$$

where $W(t)$ is Brownian motion, σ is the volatility, and r is the constant risk-free rate. Let $M(t)$ be the value of a money market account with $M(0) = 1$, i.e. $1 is invested initially at time 0, which earns the risk-free rate r.

   (a) Find the SDE for the relative price of the stock, Z_0, using the money market account $M(t)$ as the *numeraire*.

   (b) Let V_t be the value at time t of a contingent claim on S with payoff V_T at time T. If Z is the relative price of the contingent claim using the money market *numeraire*, then state a formula for the price of the claim at t in terms of an expectation of the payoff at T.

   (c) Explain the general principle used in the pricing formula in the previous part.

4. Consider the SDE:

$$dX = \sigma X dW, \quad X(0) = x_0$$

where $W(t)$ is Brownian motion.

   (a) Find the *strong solution* of the SDE with the given initial condition.

   (b) Construct the conditional probability density function $p(x,t;x_0)dx = \Pr[X(t) \in (x, x+dx)|X(0) = x_0]$.

   (c) What is $\mathrm{E}[X(t)|X(0) = x_0]$?

Fluid Dynamics

5. The Euler equations for the velocity vector $\vec{v}(\vec{x}, t)$ of an incompressible, inviscid fluid are

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\nabla P, \quad \nabla \cdot \vec{v} = 0$$

   (a) Obtain the equations for the vorticity $\vec{\omega} = \text{curl } \vec{v}$

   (b) Explain why in 2-dimensions the velocity can be written as the perpendicular gradient of a scalar field $\psi(x_1, x_2, t)$.

   (c) Write the vorticity equation in terms of ψ.

   (d) Show that any function $\psi(x_1, x_2)$ such that $\nabla^2 \psi = f(\psi)$ satisfies the **steady** vorticity equation.

6. Consider the Euler equations in the previous question. Prove Kelvin's circulation theorem, namely in 3-dimensions

$$\frac{d\Gamma}{dt} = 0$$

where $\Gamma = \oint_{C(t)} \vec{v} d\vec{s}$ and $\vec{v}$ is the velocity which is a function of space and time and $C(t)$ is a closed curve that moves with the fluid.

7. The speed $v(y, t)$ of a viscous fluid satisfies the *boundary layer* equation

$$\frac{\partial v}{\partial t} = \gamma \frac{\partial^2 v}{\partial y^2}$$

with

$$v(y, 0) = 0, \quad y > 0, \quad \text{at } t = 0$$
$$v(0, t) = V, \quad v(\infty, t) = 0, \quad t > 0.$$

Use the similarity variable $\eta = y/(\gamma t)^{1/2}$ to solve this problem for $v = f(\eta)$. Here γ and V are constants.

8. Consider the system of ODEs that govern the evolution of a high frequency perturbation of a steady flow $\vec{u}(\vec{x})$:

$$\frac{d\vec{k}}{dt} = -\left(\frac{\partial \vec{u}}{\partial \vec{x}}\right)^T \vec{k}$$
$$\frac{d\vec{b}}{dt} = -\left(\frac{\partial \vec{u}}{\partial \vec{x}}\right) \vec{b} + 2\left[\left(\frac{\partial \vec{u}}{\partial \vec{x}}\right) \vec{b}\right] \cdot \vec{k} \ \ \frac{\vec{k}}{|\vec{k}|^2}.$$

In the case of a 2-dimensional flow $\vec{u}$ with stream function $\psi = x_1 x_2$. Show that there exists a solution to this system such that $|\vec{b}| = e^t$. Hence explain why this flow is unstable. [$\left(\frac{\partial \vec{u}}{\partial \vec{x}}\right)$ denotes the matrix with entries $\left(\frac{\partial u_i}{\partial x_j}\right)$].