MATHEMATICAL SCIENCE PRELIMINARY EXAMINATION

Monday, May 5, 2003 1:00-4:00pm

The exam is based on questions from the areas: **Fluid Dynamics** and **Computational Finance**. There are 4 questions in each area. Each question is worth 20 points. All questions will be graded, but your score for the examination will be the sum of the scores of your best **FIVE** questions.

Use a separate answer booklet for each question, and do not put your name on the answer booklets, instead put the number that is on the envelope the exam was in. When you have completed the examination, insert all your answer booklets in the envelope provided. Then seal and print your name on the envelope.

Computational Finance

1. An asset with price $S(t)$ pays a continuous yield q and is modelled by the SDE:

$$dS = \mu S dt + \sigma S dW.$$

Here $W(t)$ is Brownian motion, μ is the real world return, and σ is the volatility. Also, assume r is the constant risk-free interest rate.

   (a) *Derive* a Black-Scholes type equation for $p(S,t)$, the price of a European put option on this asset with expiration T and strike price E. Be sure to state the final pricing problem completely.

   (b) State the SDE for the risk-neutral dynamics of the asset price described in (a).

   (c) Derive a put-call parity formula for this asset. Hint: Solve for the price of a derivative with payoff $S - E$ at T.

2. Consider a three step multiplicative binomial tree for pricing a European call option on an asset with price S today. The time step is δt and r is the constant risk-free interest rate. The multiplicative jump factors are $u > 1$ (up) and $d < 1$ (down), where $u \times d = 1$.

   (a) Find the risk-neutral probabilities q (up) in terms of u, d, and r.

   (b) Sketch the asset tree for *only* 3 steps and label the asset prices.

   (c) Compute the probability of the asset attaining the values at time step 3 in terms of q.

   (d) Give an explicit formula (in terms of q and the other parameters) for the price of the call option today, if the strike price is E and the asset price is *at-the-money* today.

3. The price of an asset at time t is given by

$$S = S_0 e^{(r-\sigma^2/2)t+\sigma X}$$

where X is normal with mean 0 and variance t. Compute E$[S]$ and Var(S).

4. The risk-neutral dynamics of an asset with price $S(t)$ is:

$$dS = rSdt + \sigma S dW$$

where $W(t)$ is Brownian motion, σ is the volatility, and r is the constant risk-free rate.

   (a) Define the forward price F of the asset at time T given that its price today (time t) is S.

   (b) Derive an SDE for for F.

   (c) Construct the conditional density function $p(F,T;S,t)$ using the SDE for F in (b).

Fluid Dynamics

5. Consider the 3 dimensional Navier-Stokes equations in cylindrical co-ordinates

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q} = -\nabla P + \nu\nabla^2\vec{q}, \quad \nabla\cdot\vec{q} = 0 \tag{1}$$

   (a) Write the equations in component form.

   (b) Determine a solution when $\nabla P = (0, 0, P_0)$, where P_0 is a constant.

6. Consider the Euler equations in the previous question, equation (1), with $\nu = 0$.

   (a) Prove that the vorticity $\vec{\omega} = \text{curl}\ \vec{q}$ satisfies

$$\frac{\partial}{\partial t}\vec{\omega} + \nabla\times(\vec{\omega}\times\vec{q}) = 0. \tag{2}$$

   (b) Let $\lambda(\vec{x}, t)$ be any scalar field such that

$$\frac{\partial}{\partial t}\lambda + (\vec{q}\cdot\nabla)\lambda = 0.$$

   Prove that

$$\left(\frac{\partial}{\partial t} + \vec{q}\cdot\nabla\right)(\vec{\omega}\cdot\nabla\lambda) = 0.$$

7. Consider the Euler equations in 2 dimensions with $\vec{x} = (x_1, x_2)$.

   (a) Show that $\vec{q} = \vec{k}\times\nabla\psi$ satisfies $\nabla\cdot\vec{q} = 0$, where $\psi = \psi(x_1, x_2, t)$ and $\vec{k}$ is the unit vector perpendicular to the 2-D plane of (x_1, x_2).

   (b) Write the vorticity equation in terms of ψ, i.e. the 2-dim version of equation (2).

   (c) Show that $\psi = \sin x_1 \sin x_2$ satisfies the vorticity equations.

8. The Rayleigh equation for $\phi(x_1)$ is

$$(u(x_1) - c)\left(\frac{d^2}{dx_1^2} - k^2\right)\phi - u''(x_1)\phi = 0$$

   with boundary conditions

$$\phi(0) = \phi(1) = 0,$$

   where $u(x_1)$ is a given function of x_1. Prove that $u'' \neq 0$ is a sufficient condition for the imaginary part of c to be zero, i.e. the Rayleigh criterion for shear flow stability.