MATHEMATICAL SCIENCE PRELIMINARY EXAMINATION

Monday, April 30, 2001 1:00-4:00pm

The Mathematical Science Preliminary examination covers the areas of Applied Optimal Control, Computational Finance, and Mathematics of Fluid Dynamics. Students elect to answer questions in **two** of these areas.

This exam is based on questions from the areas: **Applied Optimal Control** and **Computational Finance**. There are 4 questions in each area. Each question is worth 20 points. All questions will be graded, but your score for the examination will be the sum of the scores of your best **FIVE** questions.

Use a separate answer booklet for each question, and do not put your name on the answer booklets, instead put the number that is on the envelope the exam was in. When you have completed the examination, insert all your answer booklets in the envelope provided. Then seal and print your name on the envelope.

Computational Finance

1. The price of an asset $S(t)$ is modeled using geometric Brownian motion with mean μ and volatility σ.
   (a) State the stochastic differential equation (SDE) for the price $S(t)$.
   (b) Derive an SDE for $\log S(t)$
   (c) Using the SDE in (b), find the probability density function that the asset price is $S(T) = \hat{S}$ given that $S(0) = S$.
   (d) Compute $\mathrm{E}[\log S(t)]$?
2. Derive the pricing equation (Black-Scholes type) for a European put option on a non-dividend paying asset with price $S(t)$, which follows a geometric Brownian motion with mean μ and volatility σ. Assume the risk-free interest rate is r. When you have completed the derivation be sure to state completely the problem for pricing the put option.
3. Consider the price, $c(S,t)$, of a European call option on a non-dividend paying asset with exercise price E and expiration date T. Let μ be the mean return of the asset, σ be the volatility, and r be the risk-free rate.
   (a) State the Black-Scholes formula.
   (b) Derive a one term expression for the hedge ratio Δ.
   (c) Compute $\lim_{t \to T} \Delta$, if $S(T) > E$ and $\lim_{t \to T} \Delta$, if $S(T) < E$. Are the limits consistent with hedging the option?
   (d) Derive the lower bound
   $$c(S,t) \geq S - Ee^{-r(T-t)}$$
4. Derive the put-call parity formula for European options on non-dividend paying assets using the Black-Scholes equation. (Hint: Form a portfolio of puts and calls that are contained in the parity formula.)

Applied Optimal Control

5. For the deterministic linear first order dynamics,

$$\dot{x}(t) = ax(t) + bu(t), t > 0, \text{ given } x(0) = x_0 \neq 0, \ a < 0, \ b \neq 0,$$

and quadratic performance measure,

$$J[u] = \frac{1}{2} c \int_0^{t_f} u^2(t) dt, \ c > 0,$$

find the optimal state trajectory and optimal (unconstrained) control to bring the state from the initial state to the origin in t_f seconds while minimizing the functional $J[u]$ with respect to the control u, with the answer depending on the parameter set $\{x_0, t_f, a, b, c\}$. Note that the final time is *free*.

6. Derive the power rules for the Itô stochastic integration for the following Gaussian and Poisson noise integrals:

   (a) $\int_0^t W^2(s)dW(s), \ W(0) \equiv 0$, in terms of Gaussian process $W(t)$ and $\int_0^t W(s)ds$;
   (b) $\int_0^t P^2(s)dP(s), \ P(0) \equiv 0$, in terms of the Poisson process $P(t)$.

   using Itô stochastic differentiation rules generalized to Poisson processes.

7. Solve the Itô lognormal SDE for $X(t)$,

$$dX(t) = \mu(t)X(t)dt + \sigma_0 X(t)dW, \ t > 0, \ X(0) = x_0, \ \mathrm{E}[W(t)] = 0, \mathrm{Var}[W(t)] = t,$$

Using this solution show that

   (a) the expectation of the state is $\mathrm{E}[X(t)] = x_0 e^{m(t)}, \ m(t) \equiv \int_0^t \mu(t)dt$;
   (b) the corresponding *squared coefficient of variation* is $\mathrm{Var}[X(t)]/\mathrm{E}^2[X(t)] = e^{\sigma_0^2 t} - 1$.

   Recall that the Gaussian (Wiener) process $W(t)$ is normally distributed.

8. For a linear quadratic Gaussian (LQG) problem, the linear dynamics is

$$dX(t) = (aX(t) + bU(t))dt + cX(t)dW(t), \ t > 0, \ X(0) = x_0, \ a \neq 0, \ b \neq 0, \ c \neq 0,$$

where the control process $U(t)$ in unconstrained, and the quadratic criterion is

$$J[X(t), U(t)] = \frac{1}{2} S_f X^2(t_f) + \frac{1}{2} \int_t^{t_f} \left(qX^2(t) + rU^2(t) \right) dt, \ q > 0, \ r > 0, \ S_f > 0,$$

   (a) find the PDE of Stochastic Dynamic Programming for the optimal expected value:

$$v^*(x,t) = \min_u \left[\mathrm{E} \left[J[X(t), U(t)] \,|X(t) = x, U(t) = u \right] \right],$$

   and find the optimal (unconstrained) control $u^*(x,t)$ in terms of the *shadow price* $v^*_x(x,t)$;

   (b) show that this PDE of SDP formally admits a pure quadratic form solution $v^*(x,t) = \frac{1}{2} S(t) x^2$ by deriving the resulting final value problem for a Riccati equation that determines the coefficient $S(t)$ (do not solve) and find the *linear feedback control law* for $u^*(x,t)$ in terms of x, $S(t)$ and other parameters.

Sample Questions
Mathematics of Fluid Dynamics

9. Consider the Euler equations for the velocity vector $\vec{v}(\vec{x},t)$ and the pressure scalar $P(\vec{x},t)$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\nabla)\vec{v} = -\nabla P, \quad \nabla\cdot\vec{v} = 0$$

   (a) Obtain the equation for the vorticity $\vec{\omega} = \nabla\times\vec{v}$

   (b) Prove that in 2 dimensions

$$\left(\frac{\partial}{\partial t} + \vec{v}\cdot\nabla\right)\vec{\omega} = \vec{0}$$

   (c) If initially $\vec{\omega}(\vec{x},0) = \vec{0}$, show that the stream function $\psi(\vec{x})$ satisfies $\nabla^2\psi = 0$.

10. The Rayleigh equation for oscillations on a shear flow $u(y)$ is

$$\frac{d^2\phi}{dy^2} + \left(\frac{ku''}{\omega - ku} - k^2\right)\phi = 0, \;\; \phi(0) = 0, \; \phi(1) = 0.$$

Construct an energy integral to prove that a necessary condition for $\omega_I \neq 0$ is that u'' changes sign in $(0,1)$. Here $\omega = \omega_R + i\omega_I$.