Portfolio Optimization with Jump–Diffusions: Estimation of Time-Dependent Parameters and Application[1]

Floyd B. Hanson[2] John J. Westman[3]

Abstract

This paper treats jump-diffusion processes in continuous time, with emphasis on the jump-amplitude distributions, developing more appropriate models using parameter estimation for the market in one phase and then applying the resulting model to a stochastic optimal portfolio application in a second phase. The new developments are the use of uniform jump-amplitude distributions and time-varying market parameters, introducing more realism into the application model, a Log-Normal-Diffusion, Log-Uniform-Jump model.

1. Introduction

The empirical distribution of daily log-returns for actual financial instruments differ in many ways from the ideal log-normal diffusion process as assumed in the Black-Scholes model [1] and other financial models. The log-returns are the log-differences between two successive trading days, representing the logarithm of the relative size. The most significant difference is that actual log-returns exhibit occasional large jumps in value, whereas the diffusion process in Black-Scholes [1] is continuous. Another difference is that the empirical log-returns are usually negatively skewed, since the negative jumps or crashes are likely to be larger or more numerous than the positive jumps for many instruments, whereas the normal distribution associated with the diffusion process is symmetric. Thus, the coefficient of skew [2] is negative, $\eta_3 \equiv M_3/(M_2)^{1.5} < 0$, where M_2 and M_3 are the 2nd and 3rd central moments of the log-return distribution here. A third difference is that the empirical distribution is usually leptokurtic since the coefficient of kurtosis [2] satisfies $\eta_4 \equiv M_4/(M_2)^2 > 3$, where the value 3 is the normal distribution kurtosis value and M_4 is the fourth central moment. Qualitatively, this means that the tails are fatter than a normal with the same mean and standard deviation, compensated by a distribution that is also more slender about the mode (local maximum). A fourth difference is that the market exhibits time-dependence in the distributions of log-returns, so that the associated parameters are time-dependent.

In 1976, Merton [10, Chap. 9] introduced Poisson jumps with independent identically distributed random jump-amplitudes with fixed mean and variances into the Black-Scholes model, but the ability to hedge the volatilities was not very satisfactory. Kou [9] uses a jump-diffusion model with a double exponential jump-amplitude distribution with mean κ and variance 2η, having leptokurtic and negative skewness properties, although it is difficult to see the empirical justification for this distribution.

Prior to the Black-Scholes model, Merton [10, Chap. 5-6] analyzed the optimal consumption and investment portfolio with either geometric Brownian motion or Poisson noise and examined an example of constant risk-aversion utility having explicit solutions. In [10, Chap. 4], Merton also examined constant risk-aversion problems.

Hanson and Westman [5] reformulated an important external events model of Rishel [11] solely in terms of stochastic differential equations and applied it to the computation of the optimal portfolio and consumption policies problem for a portfolio of stocks and a bond. The stock prices depend on both scheduled and unscheduled jump external events. The computations were illustrated with a simple log-bi-discrete jump-amplitude model, either negative or positive jumps, such that both stochastic and quasi-deterministic jump magnitudes were estimated. In [6], they constructed a jump-diffusion model with marked Poisson jumps that had a log-normally distributed jump-amplitude and rigorously derived the density function for the log-normal-diffusion and log-normal-jump stock price log-return model. In [7], this financial model is applied to the optimal portfolio and consumption problem for a portfolio of stocks and bonds including computational results.

In this paper, the log-normal-diffusion, log-uniform-jump problem is treated. In Section 2, the jump-diffusion density is rigorously derived using a modification of our prior theorem [6]. In Section 3, the time dependent parameters for this log-return process are estimated using this theoretical density and the S&P500 Index daily closing data for the prior decade. In Section 4, the optimal portfolio and consumption policy application is presented and then solved. Concluding remarks are given in Section 5.

2. Log-Return Density for Log-Normal-Diffusion, Log-Uniform Jump

Let $S(t)$ be the price of a single financial instrument, such as a stock or mutual fund, that is governed by a Markov, geometric jump-diffusion stochastic differential equation (SDE) with time-dependent coefficients,

$$dS(t) = S(t)\left[\mu_d(t)dt + \sigma_d(t)dZ(t) + J(t)dP(t)\right] , \qquad (1)$$

with $S(0) = S_0,\ \ S(t) > 0$, where $\mu_d(t)$ is the appreciation return rate at time t, $\sigma_d(t)$ is the diffusive volatility, $Z(t)$ is is a continuous, one-dimensional Gaussian process, $P(t)$ is a discontinuous, one-dimensional standard Poisson process with jump rate $\lambda(t)$, and associated jump-amplitude $J(t)$ with log-return mean $\mu_j(t)$ and variance $\sigma_j^2(t)$. The stochastic processes $Z(t)$ and $P(t)$ are assumed to be Markov and pairwise independent. The jump-amplitude $J(t)$, given that a Poisson jump in time occurs, is also independently distributed. The stock price SDE (1) is similar in our prior work [6, 7], except that time-dependent coefficients introduce more realism here.

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[1]Work supported in part by the National Science Foundation Grant DMS-99-73231 and DMS-02-07081 at the University of Illinois at Chicago. Preprint of paper to appear in *Proceedings of 2002 Conference on Decision and Control*, pp. 1-6, Las Vegas, 9-13 December 2002.

[2]Laboratory for Advanced Computing, University of Illinois at Chicago,851 Morgan St., M/C 249, Chicago, IL 60607-7045, USA, e-mail: hanson@math.uic.edu

[3]Department of Mathematics and Statistics, Miami University, Oxford, OH 45056, USA and Department of Mathematics, University of California, Los Angeles, e-mail: westmanjj@muohio.edu

The continuous, differential diffusion process $dZ(t)$ is standard, so has zero mean and dt variance. The symbolic notation for the discontinuous space-time jump process, $J(t)dP(t)$, is better defined in terms of the Poisson random measure, $\mathcal{P}(dt, dq)$, by the stochastic integral, $J(t)dP(t) = \int_{\mathcal{Q}} \widehat{J}(t;q)\mathcal{P}(dt, dq)$, where $Q = q$ is the Poisson spatial mark variable for the jump amplitude process, and $\widehat{J}(t;q)$ is the kernel of the Poisson operator $J(t)$, such that $-1 < \widehat{J}(t;q) < \infty$ so that a single jump does not make the underlying non-positive. The infinitesimal moments of the jump process are $\mathrm{E}[J(t)dP(t)] = \lambda(t)dt \int_{\mathcal{Q}} \widehat{J}(t;q)\phi_Q(q;t)dq$ and $\mathrm{Var}[J(t)dP(t)] = \lambda(t)dt \int_{\mathcal{Q}} \widehat{J}^2(t;q)\phi_Q(q;t)dq$, neglecting $O^2(dt)$ here, where $\phi_Q(q;t)$ is the Poisson amplitude mark density. The differential Poisson process is a counting process with the probability of the jump count given by the usual Poisson distribution, $p_k(\lambda(t)dt) = \exp(-\lambda(t)dt)(\lambda(t)dt)^k/k!$, $k = 0, 1, 2, \ldots$, with parameter $\lambda(t)dt > 0$.

Since the stock price process is geometric, the common multiplicative factor of $S(t)$ can be transformed away yielding the SDE of the stock price log-return using the stochastic chain rule for Markov processes in continuous time,

$$d[\ln(S(t))] = \mu_{ld}(t)dt + \sigma_d(t)dZ(t) + \ln(1 + J(t))dP(t), \quad (2)$$

where $\mu_{ld}(t) \equiv \mu_d(t) - \sigma_d^2(t)/2$ is the log-diffusion drift and $\ln(1 + \widehat{J}(t;q))$ the stock log-return jump-amplitude is the logarithm of the relative post-jump-amplitude. This log-return SDE (2) will be the model that will used for comparison to the S&P500 log-returns. Since $\widehat{J}(t;q) > -1$, it is convenient to select the mark process to be the jump-amplitude random variable, $Q = \ln\left(1 + \widehat{J}(t;Q)\right)$, on the mark space $\mathcal{Q} = (-\infty, +\infty)$. Though this is a convenient mark selection, it implies the time-independence of the jump-amplitude, so $\widehat{J}(t;Q) = \widehat{J}_0(Q)$ or $J(t) = J_0$. Since market jumps are rare and the tails are relatively flat, a reasonable approximation is uniform jump-amplitude distribution with density $\phi_Q(q;t)$ on the finite, time-dependent mark interval $[Q_a(t), Q_b(t)]$,

$$\phi_Q(q;t) \equiv \frac{H(Q_b(t) - q) - H(Q_a(t) - q)}{Q_b(t) - Q_a(t)}, \quad (3)$$

where $H(x)$ is the Heaviside, unit step function. The density $\phi_Q(q;t)$ yields a mean $\mathrm{E}_Q[Q] = \mu_j(t) = (Q_b(t) + Q_a(t))/2$ and variance $\mathrm{Var}_Q[Q] = \sigma_j^2(t) = (Q_b(t) - Q_a(t))^2/12$, which define the basic log-return jump amplitude moments. It is assumed that $Q_a(t) < 0 < Q_b(t)$, to make sure that both negative and positive jumps are represented, which was a problem for the log-normal jump-amplitude distribution in [7]. The uniform distribution is treated as time-dependent in this paper, so $Q_a(t)$, $Q_b(t)$, $\mu_j(t)$ and $\sigma_j^2(t)$ all depend on t. The difficulty in separating out the small jumps about the mode or maximum of real market distributions is explained by the fact that a diffusion approximation for small marks can be used for the jump process that will be indistinguishable from the continuous Gaussian process anyway.

The basic moments of the stock log-return differential are

$$M_1^{(\mathrm{jd})} \equiv \mathrm{E}[d[\ln(S(t))]] = (\mu_{ld}(t) + \lambda(t)\mu_j(t))dt, \quad (4)$$

$$\begin{aligned} M_2^{(\mathrm{jd})} &\equiv \mathrm{Var}[d[\ln(S(t))]] \\ &= \left(\sigma_d^2(t) + \lambda(t)\left(\sigma_j^2(t)(1 + \lambda(t)dt)\mu_j^2(t)\right)\right)dt, \end{aligned} \quad (5)$$

where the $O^2(dt)$ term has been retained in the variance, rather than being neglected as usual, since the discrete return time, $dt = \Delta t$, the daily fraction of one trading year (about 250 days), will be small, but not negligible.

The log-normal-diffusion, log-uniform-jump density can be found by basic probabilistic methods following a slight modification for time-dependent coefficients of constant coefficient theorem found our paper [6],

Theorem: The probability density for the log-normal-diffusion, log-uniform-jump amplitude log-return differential $d[\ln(S(t))]$ specified in the SDE (2) with time-dependent coefficients is given by

$$\begin{aligned} \phi_{d\ln(S(t))}(x) &= p_0(\lambda(t)dt)\phi^{(n)}\left(x;\mu_{ld}(t)dt, \sigma_d^2(t)dt\right) \\ &+ \textstyle\sum_{k=1}^{\infty} \frac{p_k(\lambda(t)dt)}{k(Q_b(t) - Q_a(t))} \\ &\cdot \Big[\Phi^{(n)}\left(kQ_b(t) - x + \mu_{ld}(t)dt; 0, \sigma_d^2(t)dt\right) \\ &- \Phi^{(n)}\left(kQ_a(t) - x + \mu_{ld}(t)dt; 0, \sigma_d^2(t)dt\right)\Big], \end{aligned} \quad (6)$$

$-\infty < z < +\infty$, where $p_k(\lambda(t)dt)$ is the Poisson distribution and the normal distribution with mean $\mu_{ld}dtr$ and variance $\sigma_d^2 dt$ is

$$\Phi^{(n)}(x;\mu_{ld}dt, \sigma_d^2 dt) = \int_{-\infty}^{x} \phi^{(n)}(y;\mu_{ld}dt, \sigma_d^2 dt)dy$$

associated with $d\ln(S(t))$, the diffusion part of the log-return process,

$$\phi_{\mu_{ld}dt + \sigma_d dZ(t)}(x) = \phi^{(n)}(x;\mu_{ld}(t)dt, \sigma_d^2(t)dt) .$$

The proof, which is only briefly sketched here, follows from the density of a triad of independent random variables, $\xi + \eta \cdot \zeta$ given the densities of the three component processes ξ, η, and ζ. Here, (1) $\xi = \mu_{ld}(t)dt + \sigma_d(t)dZ(t)$ is the log-normal plus log-drift diffusion process, (2) $\eta = Q = \ln(1 + \widehat{J}_0(Q))$ is the log-uniform jump-amplitude, and (3) $\zeta = dP(t)$ is the differential Poisson process. The density of a sum of independent random variables, as in the sum operation of $\xi + (\eta \cdot \zeta)$, is very well-known and is given by a convolution of densities $\phi_{\xi+\eta\zeta}(z) = \int_{-\infty}^{+\infty} \phi_\xi(z-y)\phi_{\eta\zeta}(y)dy$ (see Feller [3]). However, the distribution of the product of two random variables $\eta \cdot \zeta$ is not so well-known [6] and has the density,

$$\begin{aligned} \phi_{\eta\zeta}(x) &= p_0(\lambda(t)dt)\delta(x) \\ &+ \textstyle\sum_{k=1}^{\infty} \frac{p_k(\lambda(t)dt)[H(Q_b(t) - x/k) - H(Q_a(t) - x/k)]}{k(Q_b(t) - Q_a(t))}, \end{aligned} \quad (7)$$

for the log-uniform-jump process. The probabilistic mass at $x = 0$, represented by the Dirac $\delta(x)$ and corresponds to the zero jump event case. Finally, applying the convolution formula for density of the sum $\xi + (\eta\zeta)$ leads to the density for the random variable triad $\xi + \eta\zeta$ given in (6) of the theorem.

Using the log-normal jump-diffusion log-return density in (6), the third and fourth central moments with finite return time $dt = \Delta t$ are computed, for later use for skew and kurtosis coefficients, respectively, yielding the jump-diffusion higher moments [6],

$$\begin{aligned} M_3^{(\mathrm{jd})} &\equiv \mathrm{E}\left[\left(d[\ln(S(t))] - M_1^{(\mathrm{jd})}\right)^3\right] \\ &= 6\mu_j(t)(\lambda(t)dt)^2\sigma_j^2(t) + (3\mu_j(t)\sigma_j^2(t) + \mu_j^3(t))\lambda(t)dt, \end{aligned} \quad (8)$$

$$\begin{aligned} M_4^{(\mathrm{jd})} &\equiv \mathrm{E}\left[\left(d[\ln(S(t))] - M_1^{(\mathrm{jd})}\right)^4\right] \\ &= 3(\sigma_j^2(t))^2(\lambda(t)dt)^4 + (6\mu_j^2(t)\sigma_j^2(t) \\ &+ 18(\sigma_j^2(t))^2)(\lambda(t)dt)^3 + (3\mu_j^4(t) + 30\mu_j^2(t)\sigma_j^2(t) \\ &+ 21(\sigma_j^2(t))^2 + 6\sigma_j^2(t)dt\sigma_j^2(t))(\lambda(t)dt)^2 \\ &+ (\mu_j^4(t) + 6\sigma_j^2(t)dt\sigma_j^2(t) + 6\mu_j^2(t)\sigma_j^2(t) \\ &+ 6\mu_j^2(t)\sigma_j^2(t)dt + 3(\sigma_j^2(t))^2)\lambda dt + 3(\sigma_j^2(t))^2 dt^2 . \end{aligned} \quad (9)$$

3. Jump-Diffusion Parameter Estimation

Given the log-normal-diffusion, log-uniform-jump density (6), it is necessary to fit this theoretical model to realistic empirical data to estimate the parameters of the log-return model (2) for $d[\ln(S(t))]$. For realistic empirical data, the daily closings of the S&P500 Index during the decade from 1992 to 2001 are used from data available on-line [13]. The data consists of $n^{(\mathrm{sp})} = 2522$ daily closings. The S&P500 data can be viewed as an example of one large mutual fund rather than a single stock. The data has been transformed into the discrete analog of the continuous log-return, i.e., into changes in the natural logarithm of the index closings, $\Delta[\ln(SP_i)] \equiv \ln(SP_{i+1}) - \ln(SP_i)$ for $i = 1, \ldots, n^{(\mathrm{sp})} - 1$ daily closing pairs. For the decade, the mean is $M_1^{(\mathrm{sp})} \simeq 4.015 \times 10^{-4}$ and the variance is $M_2^{(\mathrm{sp})} \simeq 9.874 \times 10^{-5}$, the coefficient of skewness is $\eta_3^{(\mathrm{sp})} \equiv M_3^{(\mathrm{sp})}/(M_2^{(\mathrm{sp})})^{1.5} \simeq -0.2913 < 0$, demonstrating the typical negative skewness property, and the coefficient of kurtosis is $\eta_4^{(\mathrm{sp})} \equiv M_4^{(\mathrm{sp})}/(M_2^{(\mathrm{sp})})^2 \simeq 7.804 > 3$, demonstrating the typical leptokurtic behavior of many real markets.

The S&P500 log-returns, $\Delta[\ln(SP_i)]$ for $i = 1 : n^{(\mathrm{sp})}$ decade data points, are partitioned into 10 yearly data sets, $\Delta[\ln(SP^{(\mathrm{spy})}_{j_y,k})]$ for $k = 1 : n^{(\mathrm{sp})}_{y,j_y}$ yearly data points for $j_y = 1 : 10$ years, where $\sum_{j_y=1}^{10} n^{(\mathrm{sp})}_{y,j_y} = n^{(\mathrm{sp})}$. For each of these yearly sets, the parameter estimation objective is to find the least sum of squares of the deviation between the empirical S&P500 log-return histograms for the year and the analogous theoretical log-normal-diffusion, log-uniform-jump distribution histogram based upon the same bin structure. Since jumps are rare, 100 centered bins within the log-return domain $[x_a, x_b]$ were used. Since the most extreme log-returns are the same as the most extreme jumps, the log-return domain is selected to coincide with the time-dependent uniform distribution domain, i.e., $[x_a(t), x_b(t)] = [Q_a(t), Q_b(t)]$, both dimensionless, where $Q_a(t) = \min_k(\Delta[\ln(SP^{(\mathrm{spy})}_{j_y,k})])$ and $Q_b(t) = \max_k(\Delta[\ln(SP^{(\mathrm{spy})}_{j_y,k})])$ with $t = T_{j_y} = \mathrm{Year}_{j_y} + 0.5$, say, assigning the yearly value to the mid-year with steps of $dt = \Delta T_{j_y}$, for each $j_y = 1 : 10$. For a given $t = T_{j_y}$ year, fixed $[Q_a(t), Q_b(t)]$ implies fixed uniform distribution parameters $\mu_j(t) = (Q_b(t) + Q_a(t))/2$ and $\sigma_j^2(t) = (Q_b(t) - Q_a(t))^2/12$. However, the Poisson jump rate $\lambda(t)$ is still a free parameter for the jump component of the log-return process. Further to keep the number of free parameters as small as practical, we require that the mean and variances of the yearly log-returns be the same for both empirical and theoretical distributions, i.e.,

$$M^{(\mathrm{spy})}_{1,j_y} \equiv \mathrm{Mean}_{k=1}^{n^{(\mathrm{sp})}_{y,j_y}} \left[\Delta\left[\ln\left(SP^{(\mathrm{spy})}_{j_y,k}\right)\right]\right] = M^{(\mathrm{jdy})}_{1,j_y}$$

using (4) and

$$M^{(\mathrm{spy})}_{2,j_y} \equiv \mathrm{Var}_{k=1}^{n^{(\mathrm{sp})}_{y,j_y}} \left[\Delta\left[\ln\left(SP^{(\mathrm{spy})}_{j_y,k}\right)\right]\right] = M^{(\mathrm{jdy})}_{2,j_y}$$

using (5), for each $j_y = 1 : 10$ years. This, in turn, implies constraints on the log-diffusion parameters,

$$\mu_{ld,j_y} = \left(M^{(\mathrm{spy})}_{1,j_y} - (\lambda\, dt \mu_j)_{j_y}\right)/\Delta T_{j_y}\ , \qquad (10)$$

$$\sigma^2_{d,j_y} = \left(M^{(\mathrm{spy})}_{2,j_y} - (\lambda dt((1+\lambda dt)\sigma_j^2 + \mu_j^2)_{j_y})\right)/\Delta T_{j_y}\ , \qquad (11)$$

with $\sigma^2_{d,j_y} > 0$ for each $j_y = 1 : 10$ years. Of the six parameters $\{\mu_{ld,j_y}, \sigma^2_{d,j_y}, \mu_{j,j_y}, \sigma^2_{j,j_y}, \lambda_{j_y}, \Delta T_{j_y}\}$, needed for each year j_y to specify the jump-diffusion log-return distribution, only the jump rate λ_{j_y} needs to be estimated by least squares. The time step $dt = \Delta T_{j_y}$ is the reciprocal of the number of trading days per year, close to 250 days, but varies a little for $j_y = 1 : 10$ and has values lying in the range, [0.003936, 0.004050], used here for parameter estimation.

Thus, we have a one dimensional global minimization problem for a highly complex discretized jump-diffusion density function (2). The analytical complexity indicates that a general global optimization method that does not require derivatives would be useful. For this purpose, such a method, *Golden Super Finder (GSF)* [8], was developed for [7] and implemented in MATLAB™, since simple techniques are desirable in financial engineering. The *GSF* method is an extensive modification to the Golden Section Search method [4], extended to multi-dimensions and allowing search beyond the initial hyper-cube domain by including the endpoints in the local optimization test with the two golden section interior points per dimension, moving rather than shrinking the hypercube when the local optimum is at an edge or corner. The method, as a general method, is slow, but systematically moves the search until the uni-modal optimum is found at a interior point and then approaches the optimum if within the original search bounds. Additional constraints can be added to the objective function, such as (10,11). If the diffusion coefficient vanishes, $\sigma_d^2 \to 0^+$, then (11) implies a maximum jump count constraint, $\max[\lambda \cdot dt] = 0.5(\sqrt{((\sigma_j^2 + \mu_j^2)^2 + 4\sigma_j^2 \cdot M_2)} - (\sigma_j^2 + \mu_j^2))/\sigma_j^2$. An additional compatibility constraint, $\sigma_j(t) > 0$, does not need enforcement as long as $Q_a(t) < Q_b(t)$ and is not violated here.

The jump-diffusion estimated parameter results in this log-normal-diffusion, log-uniform-jump amplitude case are summarized in Table 1. The jump rate estimates and their variability are summarized in Table 2. A hybrid value-position stopping criterion with a tolerance, $tol = 5.e{-}3$ was used, and all yearly iterations converged in at most 13 iterations each, out of a maximum limit of 20, except for the year 1999* which exhibited little evidence of the long and flat tails of other years, with a limiting behavior indicating a zero jump rate value, $\lambda_8 \simeq 2.52e-4 \simeq 0.0$ for $\mathrm{Year}_8 = 1999^*$ when $j_y = 8$.

Table 1: Summary of yearly coefficients for Log-Normal-Diffusion, Log-Uniform-Jump estimated parameters by least squares (variance of deviation between S&P500 and jump-diffusion histograms) with respect to the variable λdt given $dt = \Delta T_{j_y}$ and constraints mentioned in the text.

Year_{j_y}	μ_{d,j_y}	σ_{d,j_y}	μ_{j,j_y}	σ_{j,j_y}	λ_{j_y}
1992	4.1e-2	7.3e-2	-1.6e-3	9.9e-3	36.
1993	6.7e-2	7.0e-2	-2.6e-3	1.3e-2	15.
1994	-1.5e-2	7.6e-2	-9.1e-4	1.3e-2	22.
1995	3.0e-1	5.8e-2	1.5e-3	9.9e-3	25.
1996	1.7e-1	9.7e-2	-6.0e-3	1.5e-2	17.
1997	2.8e-1	1.5e-1	-1.1e-2	3.5e-2	7.1
1998	2.2e-1	1.5e-1	-1.0e-2	3.5e-2	14.
1999*	1.9e-1	1.8e-1	3.1e-3	1.8e-2	2.5e-4*
2000	-1.2e-1	1.9e-1	-6.8e-3	3.1e-2	14.
2001	-1.1e-1	1.8e-1	-7.9e-4	2.9e-2	15.

In Figure 1 a sample comparison can be made of the empirical S&P500 histogram on the left for the relatively noisy year of 2000 with the corresponding theoretical jump-diffusion histogram on the right using the fitted, optimized parameters and the same number of

Table 2: Summary of final optimal search yearly positions and values, including $\lambda \cdot dt$ maximal constraint and final iteration count. The $\mathrm{Var[Deviation]}$ are the least squares approximation at stopping. For 1999*, the stopping was due to reaching the maximum number of 20 iterations.

Year_{j_y}	$(\lambda dt)_{j_y}$	$\max[\lambda dt]_{j_y}$	Var[Dev]	max[Iter]
1992	1.4e-1	2.9e-1	2.6	12
1993	5.9e-2	1.6e-1	1.9	12
1994	8.8e-2	2.0e-1	2.4	12
1995	10.e-2	2.0e-1	2.9	11
1996	6.8e-2	1.9e-1	2.8	12
1997	2.8e-2	9.0e-2	2.8	13
1998	5.7e-2	1.1e-1	1.4	12
1999*	1.0e-6*	2.9e-1	2.4	20*
2000	5.5e-2	1.7e-1	3.2	13
2001	6.2e-2	1.9e-1	3.2	12

centered bins on the domain. The jump-diffusion histogram is a very idealized version of the empirical distribution, with the asymmetry of the tails clearly illustrated, noting that the years 1997-present are more noisier than the quieter years from 1992-1995. The histogram for the yearly empirical data on the left side of Fig. 1 suggests that it may take more than a year to develop a more typical market log-return distribution.

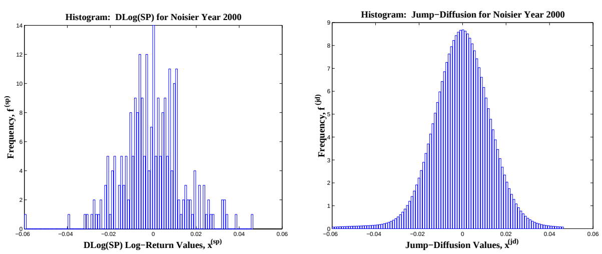

Figure 1: Comparison for the relatively noisier year 2000 of the empirical S&P500 histogram on the left with the corresponding fitted theoretical jump-diffusion histogram on the right, using 100 bins. Note that the scale on the left is 56% larger than on the right since the left figure has a unusually high peak with a count of 14 while the figure on the right has a peak count of less than 9.

For reference, the summaries of the coefficients of skewness and kurtosis are given in Table 3 for both the estimated theoretical jump-diffusion model and the empirical S&P500 data to facilitate comparison. Note that the jump-diffusion kurtosis values $\eta_4^{(\mathrm{jd})}$ are mostly double the empirical values except in 1997 and the nearly normal or non-typical case in 1999*. The S&P500 value $\eta_4^{(\mathrm{sp})} \simeq 2.85$, being *platykurtosis* since $\eta_4 < 3$, according to [2], confirming the non-typical results of the parameter estimation procedure for 1999*. The 1992 year is also nearly neutral in kurtosis being close to the *normal* value of three. The discrepancy between the estimated theoretical and observed data for kurtosis is likely due to the relative smallness of the yearly sample as well as the bin size and fixed the yearly uniform domain. Further, the concept that the market data is usually leptokurtic apparently refers to long term data and not to shorter term data.

The main purpose of this parameter estimation has been to have an estimate of the parameter time-dependence. Hence, we used the

Table 3: Summary of yearly coefficients of skewness, η_3, and kurtosis, η_4, for both the estimated theoretical including $\lambda \cdot dt$ maximal constraint and jump–diffusion (superscript (jd)) model and empirical S&P500 (superscript (sp)) decade data.

Year_{j_y}	$\eta_{3,j_y}^{(\mathrm{jd})}$	$\eta_{3,j_y}^{(\mathrm{sp})}$	$\eta_{4,j_y}^{(\mathrm{jd})}$	$\eta_{4,j_y}^{(\mathrm{sp})}$
1992	-3.9e-1	5.9e-2	8.9	3.2
1993	-5.1e-1	-1.8e-1	10.	5.4
1994	-1.9e-1	-3.0e-1	10.	4.3
1995	4.4e-1	-8.1e-2	11.	4.0
1996	-7.6e-1	-6.0e-1	8.7	4.7
1997	-7.9e-1	-6.9e-1	13.	9.5
1998	-1.2e+0	-6.2e-1	18.	7.7
1999*	2.1e-6	6.8e-2	3.0*	2.9
2000	-4.3e-1	-1.4e-2	8.5	4.3
2001	-5.6e-2	3.1e-2	8.3	4.5

polynomial fitting commands, *polyfit* and *polyval* of MATLAB™ to fit the jump-diffusion parameters in time using the better-posed standardized variables for middle year variables centered about the mean year and relative to the standard deviation of the decade of years, i.e., in terms of $\tau_s = (\mathrm{Year}_{j_y} - \mathrm{Mean[Year]})/\mathrm{StdDev[Year]}$. The quadratic polynomial fitting seems to be somewhat better judging from the standard deviation of the parameter value from the polynomial model value.

4. Application to Optimal Portfolio and Consumption Policies

Consider a portfolio consisting of a riskless asset, called a bond, with price $B(t)$ dollars at time t years, and a risky asset, called a stock, with price $S(t)$ at time t. Let the fractions of instantaneous portfolio change be $U_0(t)$ for the bond and $U_1(t)$ for the stock, so that the total satisfies $U_0(t) + U_1(t) = 1$. The bond price process is deterministic exponential,

$$dB(t) = r(t)B(t)dt \ , \quad B(0) = B_0 \ . \tag{12}$$

where $r(t)$ is the bond rate of interest at time t. The stock price $S(t)$ has been given in (1). The portfolio wealth process changes due to changes in the portfolio fractions less the instantaneous consumption of wealth $C(t)dt$,

$$\begin{aligned} dW(t) = W(t)\,[r(t)dt + U_1(t)\,\{(\mu_d(t) - r(t))dt \\ +\sigma_d(t)dZ(t) + J_0 dP(t)\}] - C(t)dt \ , \end{aligned} \tag{13}$$

such that $W(t) \geq 0$ and that the consumption rate is constrained relative to wealth $0 \leq C(t) \leq C_{\max}^{(0)} W(t)$, the stock fraction is bounded by fixed constants, $U_{\min}^{(0)} \leq U_1(t) \leq U_{\max}^{(0)}$, so borrowing and short-selling is permissible, and $U_0(t) = 1 - U_1(t)$ has been eliminated [7].

The investor's portfolio objective is to maximize the conditional, expected current value of the discounted utility $\mathcal{U}_f(w)$ of terminal wealth at the end of the investment terminal time T and the discounted utility of instantaneous consumption $\mathcal{U}(c)$, i.e.,

$$\begin{aligned} v^*(t,w) \quad = \quad & \max_{\{u,c\}[t,T)} \left[E\left[e^{-\beta(T-t)} \mathcal{U}_f(W(T)) \right.\right. \\ & \left.\left. + \int_t^T e^{-\beta(\tau - t)} \mathcal{U}(C(\tau))\, d\tau \right| \mathcal{C} \right] \right] \ , \end{aligned} \tag{14}$$

conditioned on the state-control set $\mathcal{C} = \{W(t) = w, U_1(t) = u, C(t) = c\}$, where $0 \leq t < T$, and $\beta(t)$ is the discount rate at time t. Thus, the instantaneous consumption $c = C(t)$ and stock portfolio fraction $u = U_1(t)$ serve as control variables, while the wealth $w = W(t)$ is the single state variable. Eq. (14) is subject to zero wealth absorbing natural boundary condition (avoids arbitrage [10, Chap. 6]),

$$v^*(t, 0^+) \quad = \quad \mathcal{U}_f(0)e^{-\beta(T-t)} + \mathcal{U}(0)\frac{(1-e^{-\beta(T-t)})}{\beta(T-t)} \tag{15}$$

and since the consumption must be zero when the wealth is zero. The terminal wealth condition $v^*(T, w) = \mathcal{U}_f(w)$, must also be satisfied. Assuming the $v^*(t, w)$ is continuously differentiable in t and twice continuously differentiable in w, then the stochastic dynamic programming equation (see [7]) follows from an application of the (Itô) stochastic chain rule to the principle of optimality,

$$\begin{aligned} 0 &= v_t^*(t,w) - \beta(t)v^*(t,w) + \mathcal{U}(c^*(t,w)) \\ &\quad + [(r(t) + (\mu_d(t) - r(t))u^*(t,w))w \\ &\quad - c^*(t,w)]\, v_w^*(t,w) + \tfrac{1}{2}\sigma_d^2(t)(u^*)^2(t,w)w^2 v_{ww}^*(t,w) \\ &\quad + \tfrac{\lambda(t)}{Q_b(t)-Q_a(t)} \int_{Q_a(t)}^{Q_b(t)} \\ &\quad \cdot \left[v^*(t,(1+\widehat{J}_0(q)u^*(t,w))w) - v^*(t,w)\right]\, dq\ , \end{aligned} \tag{16}$$

where $u^* = u^*(t,w) \in [U_{\min}^{(0)}, U_{\max}^{(0)}]$ and $c^* = c^*(t,w) \in [0, C_{\max}^{(0)} w]$ are the optimal controls if they exist, while $v_w^*(t,w)$ and $v_{ww}^*(t,w)$ are the partial derivatives with respect to wealth w when $0 \le t < T$. Non-negativity of wealth implies an additional consistency condition for the control since the jump in wealth argument $(1 + \widehat{J}_0(q)u^*)w$ requires $1 + \widehat{J}_0(q)u \ge 0$ on $Q_a(t) \le q \le Q_b(t)$ with $\widehat{J}_0(q) = e^q - 1$, then $-\min_t[1/(e^{Q_b(t)} - 1)] \le u \le \min_t[1/(1 - e^{Q_a(t)})]$ is required, taking the worst case scenario to *avoid the jump that wipes out the investor's wealth*. For negative jumps, $Q_a(t) \in [-0.7113, -0.01874]$ for the S&P500 decade data, so $\min_t[Q_a(t)] = -0.07113$, suggesting that we take $U_{\max}^{(0)} \simeq 10 < 1/(1 - \exp(\min_t[Q_a(t)])) \simeq 14.57$, say. For positive jumps, $Q_b(t) \in [0.01543, 0.04990]$, so $\max_t[Q_b(t)] = 0.04990$, suggesting that we take $U_{\min}^{(0)} \simeq -12 > 1/(\exp(\max_t[Q_b(t)])) \simeq -19.54$, say. In absence of control constraints, then the maximum controls are the regular controls $u_{\text{reg}}(t,w)$ and $c_{\text{reg}}(t,w)$, which are given implicitly, provided they are attainable and there is sufficient differentiability in c and u, by the dual critical conditions,

$$\mathcal{U}'(c_{\text{reg}}(t,w)) = v_w^*(t,w)\ , \tag{17}$$

$$\begin{aligned} &\sigma_d^2(t)w^2 v_{ww}^*(t,w)u_{\text{reg}}(t,w) = \\ &\quad -(\mu_d(t) - r(t))wv_w^*(t,w) \\ &\quad - \tfrac{\lambda(t)w}{Q_b(t)-Q_a(t)} \int_{Q_a(t)}^{Q_b(t)} \widehat{J}_0(q)v_w^*(t,(1 \\ &\quad + \widehat{J}_0(q)u_{\text{reg}}(t,w))w)\, dq\ , \end{aligned} \tag{18}$$

for the optimal consumption and portfolio policies with respect to the terminal wealth and instantaneous consumption utilities (14). Note that (17-18) define the set of regular controls implicitly.

Assuming the investor is risk adverse, the utilities will be the Constant Relative Risk-Aversion (CRRA) power utilities [10, 5], with the same power for both wealth and consumption,

$$\mathcal{U}(x) = \mathcal{U}_f(x) = x^\gamma/\gamma\ ,\ \ x \ge 0\ ,\ \ 0 < \gamma < 1\ . \tag{19}$$

The CRRA power utilities for the optimal consumption and portfolio problem lead to a canonical reduction of the stochastic dynamic programming PDE problem to a simpler ODE problem in time, by the separation of wealth and time dependence,

$$v^*(t,w) = \mathcal{U}(w)v_0(t), \tag{20}$$

where only the time function $v_0(t)$ is to be determined. The regular consumption control is a linear function of the wealth,

$$c_{\text{reg}}(t,w) \equiv w \cdot c_{\text{reg}}^{(0)}(t) = w/v_0^{1/(1-\gamma)}(t), \tag{21}$$

using (17) and $\mathcal{U}'(x) = x^{\gamma-1}$ using (19). The regular stock fraction u is a wealth independent control, but is given in implicit form:

$$\begin{aligned} u_{\text{reg}}(t,w) &= u_{\text{reg}}^{(0)}(t) \\ &= \tfrac{1}{(1-\gamma)\sigma_d^2(t)}\left[\mu_d(t) - r(t) + \lambda(t)I_1\left(u_{\text{reg}}^{(0)}(t)\right)\right], \\ I_1(u) &= \tfrac{1}{Q_b(t)-Q_a(t)} \int_{Q_a(t)}^{Q_b(t)} \widehat{J}_0(q)\left(1 + \widehat{J}_0(q)u\right)^{\gamma-1} dq, \end{aligned} \tag{22}$$

The wealth independent property of the regular stock fraction is essential for the separability of the optimal value function (20). Since (22) only defines $u_{\text{reg}}^{(0)}(t)$ implicitly in fixed point form, $u_{\text{reg}}^{(0)}(t)$ must be found by an iteration such as Newton's method, while the our Gauss-Statistics quadrature [12] can be used for jump integrals (see [7]). The optimal controls, when there are constraints, are given in piecewise form as $c^*(t,w)/w = c_0^*(t) = \max[\min[c_{\text{reg}}^{(0)}(t), C_{\max}^{(0)}], 0]$, provided $w > 0$, and $u^*(t,w) = u_0^*(t) = \max[\min[u_{\text{reg}}^{(0)}(t), U_{\max}^{(0)}], U_{\min}^{(0)}]$, is independent of w along with $u_{\text{reg}}^{(0)}(t)$. Substitution of the separable power solution (20) and the regular controls (21-22) into the stochastic dynamic programming equation (16), leads to an apparent Bernoulli type ODE,

$$0 = v_0'(t) + (1-\gamma)\left(g_1(t, u_0^*(t))v_0(t) + g_2(t)v_0^{\frac{\gamma}{\gamma-1}}(t)\right), \tag{23}$$

$$\begin{aligned} g_1(t,u) &\equiv \tfrac{1}{1-\gamma}\left[-\beta(t) + \gamma\left(r(t) + u(\mu_d(t) - r(t))\right)\right. \\ &\quad \left. - \tfrac{\gamma(1-\gamma)}{2}\sigma_d^2(t)u^2 + \lambda(t)(I_2(t,u) - 1)\right]\ , \\ g_2(t) &\equiv \frac{1}{1-\gamma}\left[\left(\frac{c_0^*(t)}{c_{\text{reg}}^{(0)}(t)}\right)^\gamma - \gamma\left(\frac{c_0^*(t)}{c_{\text{reg}}^{(0)}(t)}\right)\right], \\ I_2(t,u) &\equiv \frac{1}{Q_b(t) - Q_a(t)} \int_{Q_a(t)}^{Q_b(t)} \left(1 + \widehat{J}_0(q)u\right)^\gamma\ dq\ , \end{aligned}$$

for $0 \le t < T$. The coupling of $v_0(t)$ to the time dependent part of the consumption term $c_{\text{reg}}^{(0)}(t)$ in $g_2(t)$ and the relationship of $c_{\text{reg}}^{(0)}(t)$ to $v_0(t)$ in (21) means that the differential equation (23) is implicitly highly nonlinear and thus (23) is only of Bernoulli type formally. The apparent Bernoulli equation (23) can be transformed to a apparent linear differential equation by using $\theta(t) = v_0^{1/(1-\gamma)}(t)$, to obtain, $0 = \theta'(t) + g_1(t, u_0^*)\theta(t) + g_2(t)$, whose general solution can be inverse transformed to the general solution for the separated time function,

$$\begin{aligned} v_0(t) = \theta^{1-\gamma}(t) &= \left[e^{-g_1(t,u_0^*(t))(T-t)}\,(1\right. \\ &\quad \left. + \int_t^T g_2(\tau)e^{g_1(t,u_0^*(t))(T-\tau)}\ d\tau)\right]^{1-\gamma}\ , \end{aligned} \tag{24}$$

given implicitly.

In order to illustrate this stochastic application, a computational approximation of the solution is presented. The main computational changes from the procedure used in [7] are that the jump-amplitude distribution is now uniform and the portfolio parameters as well as the jump-amplitude distribution are time-dependent. The parameter time-dependence is approximated by quadratic interpolation over the decade from 1992-2001. The terminal time is taken to be $T = 11$, one year beyond this decade. For this numerical study, the economic rates are taken to constant, so the bond interest rate is $r = 5.75\%$

and the time-discount rate is $\beta = 5.25\%$. The portfolio stock fraction constraints are $[U^{(0)}_{\min}, U^{(0)}_{\max}] = [-12, 10]$ and the $C^{(0)}_{\max} = 0.75$ for consumption relative to wealth. In Figure 2, the optimal portfolio stock fraction $u^*(t)$ is displayed. The portfolio policy is not monotonic in time and the minimum control constraint at $U^{(0)}_{\min} = -12$ is active during the first half year in $t \in [0, T]$, while the maximum constraint is not activated since $u^*(t)$ remains significantly below that constraint. The $u^*(t)$ non-monotonic behavior is very interesting compared to the constant behavior in the constant parameter model in [7]. Likely the stock fraction grew initially due to the early relatively quiet period, then peaked at the beginning of the fourth year (1996 in the S&P500 data) as the market became noisier and continued to decline due to the final relatively noisier period. In

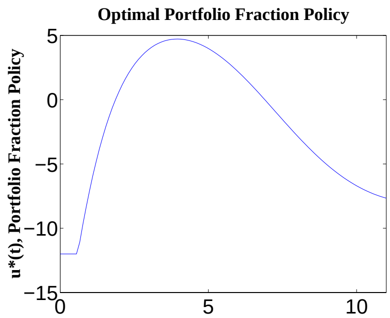

Figure 2: Optimal portfolio stock fraction policy $u^*(t)$ on $t \in [0, 11]$ subject to the control constraint set $[U^{(0)}_{\min}, U^{(0)}_{\max}] = [-12, 10]$.

Figure 3 on the left, the optimal, expected, discounted utility of terminal wealth and cumulative consumption, $v^*(t, w)$, is displayed in three dimensions. The behavior of $v^*(t, w)$ for fixed time t reflects the CRRA utility of function $\mathcal{U}(w)$ template of the separable canonical solution form in (20), while the decay in time toward the final time $T = 11$ and final value $v^*(T, w) = 0$ for fixed wealth w derives from the separable time function $v_0(t)$. The optimal value function $v^*(t, w)$ results, and the following optimal consumption policy $c^*(t, w)$ results in Fig. 3 on the right, in this computational example are qualitatively similar to that of the time-independent parameter case in the [7] computational results.

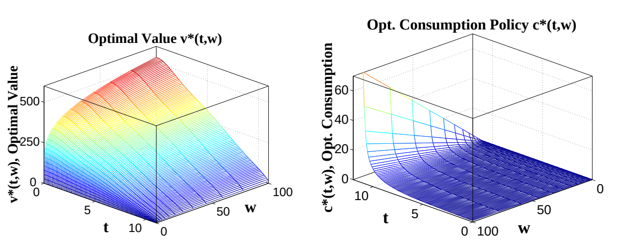

Figure 3: Optimal portfolio value $v^*(t, w)$ on the left and optimal consumption policy $c^*(t, w)$ on the right for $(t, w) \in [0, 11] \times [0, 100]$.

5. Conclusions

The main contributions of this paper are the introduction of the uniformly distributed jump-amplitude into the jump-diffusion stock price model model and the development of time-dependent in the jump-diffusion parameters. The uniformly distributed jump-amplitude feature of the model is a reasonable assumption for rare, large jumps, crashes or buying-frenzies, when there is only a sparse population of isolated jumps in the tails of the market distribution. Additional realism in the jump-diffusion model is given by the introduction of time dependence in the distribution and in the associated parameters. Further improvements, but with greater computational complexity, would be to estimate the uniform distribution limits $[Q_a, Q_b]$ by fitting the theoretical distribution to real market distributions, using longer and overlapping partitioning of the market data to reduce the effects of small sample sizes.

References

[1] Black, F., and M. Scholes, "The Pricing of Options and Corporate Liabilities," *J. Political Economy*, vol. 81 (1973), pp. 637-659

[2] Evans, M., N. Hastings, and B. Peacock, *Statistical Distributions*, 3rd edn., John Wiley, New York, 2000.

[3] Feller, W., *An Introduction to Probability Theory and Its Application*, vol. 2, 2nd edn., John Wiley, New York, 1971.

[4] Forsythe, G. E., M. A. Malcolm and C. Moler, *Computer Methods for Mathematical Computations*, Prentice-Hall, Englewood Cliffs, NJ, 1977.

[5] Hanson,F. B., and J. J. Westman, "Optimal Consumption and Portfolio Policies for Important Jump Events: Modeling and Computational Considerations," *Proceedings of 2001 American Control Conference*, June 2001, pp. 4556–4561.

[6] Hanson, F. B., and J. J. Westman, "Stochastic Analysis of Jump–Diffusions for Financial Log–Return Processes," *Proceedings of Stochastic Theory and Control Workshop*, Springer–Verlag, New York, 2003, pp. 1–15, to appear.

[7] Hanson, F. B., and J. J. Westman, "Optimal Consumption and Portfolio Control for Jump-Diffusion Stock Process with Log-Normal Jumps," *Proceedings of 2002 American Control Conference*, May 2002, pp. 4256–4261

[8] Hanson, F. B., and J. J. Westman, "Golden Super Finder: Multidimensional Modification of Golden Section Search Unrestricted by Initial Domain," in preparation, 2002.

[9] Kou, S. G., "A Jump Diffusion Model for Option Pricing with three Properties: Leptokurtic Feature, Volatility Smile, and Analytical Tractability," *URL: http://www.ieor.columbia.columbia.edu-/∼Kou/expo.pdf*, February 2000.

[10] Merton, R. C., *Continuous-Time Finance*, Basil Blackwell, Cambridge, MA, 1990.

[11] Rishel, R., "Modeling and Portfolio Optimization for Stock Prices Dependent on External Events," *Proceedings of 38th IEEE Conference on Decision and Control*, December 1999, pp. 2788–2793.

[12] Westman, J. J., and F. B. Hanson, "Nonlinear State Dynamics: Computational Methods and Manufacturing Example," *International Journal of Control*, vol. 73 (2000), pp. 464–480.

[13] Yahoo! Finance, *Historical Quotes, S & P 500 Symbol ^SPC, URL: http://chart.yahoo.com/* , February 2001.