Comparison of Market Parameters for Jump-Diffusion Distributions Using Multinomial Maximum Likelihood Estimation

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***Abstract*— Previously, we have shown that the proper method for estimating parameters from discrete, binned stock log returns is the multinomial maximum likelihood estimation, and its performance is superior to the method of least squares. Useful formulas have been derived for the jump-diffusion distributions. Numerically, the parameter estimation can be a large scale nonlinear optimization, but we have used techniques to reduce the computation demands of multi-dimensional direct search. In this paper, three jump-diffusion models using different jump-amplitude distributions are compared. These are the normal, uniform and double-exponential. The parameters of all three models are fit to the Standard and Poor's 500 log-return market data, constrained by the data first and second moments. While the results for the skew and kurtosis moments are mixed, the uniform jump distribution has superior qualitative performance since it produces genuine fat tails that are typical of market data, whereas the others have exponentially thin tails. However, the log-normal model has a big advantage in computational time of parameter estimation compared with the others, while the double-exponential is most costly due to having one more model parameter.**

I. Introduction

Despite the great success of Black-Scholes model [2] in option pricing, this pure log-normal diffusion model fails to reflect the three empirical phenomena: (1) the large random fluctuations such as crashes and rallies; (2) the non-normal features, that is, negative skewness and leptokurtic (peakedness) behavior in the stock log-return distribution; (3) the implied volatility smile, that is, the implied volatility is not a constant as in the Black-Scholes model.

Therefore, many different models are proposed to modify the Black-Scholes model so as to represent the above three empirical phenomena. Merton [9] introduced the jump-diffusion model in financial modeling, using a Poisson process for the jump timing and a log-normal process for the jump-amplitudes to describe the market crashes and rallies. Some models are proposed to incorporate the *volatility smile*, for example, Andersen, Benzoni and Lund [1] have made elaborate estimations to fit jump-diffusion models with log-normal jump-amplitudes, stochastic volatility and other features. Some models are proposed to incorporate the asymmetric features of the stock log-return distributions. Recently, Kou [8] proposed a jump-diffusion model with a log-double-exponential process for the jump-amplitudes. Since crashes and rallies are rare events, so the Poisson process is reasonable for the timing of jumps. However, there is a problem in choosing the log-normal or log-double-exponential process for the jump-amplitudes since the exponentially small tails of the log-normal and log-double-exponential distributions are contrary to the flat and thick tails of the long time financial market log-return data. Around the near-zero peak of the log-double-exponential and the log-normal, the jumps are small, so are not qualitatively different from the continuous diffusion fluctuations. Moreover, an infinite jump domain is unrealistic, since the jumps should be bounded in a real world financial markets and an infinite domain leads to unrealistic restrictions in portfolio optimization [5].

So, Hanson and Westman [4] proposed one jump-diffusion model with log-uniform jump-amplitude. Most recently, Hanson, Westman and Zhu [7] showed that for IID simulations the binned distribution is multinomial. They estimated the market parameters for this log-uniform model by subsequent multinomial maximum likelihood method to fit financial market distributions such as the Standard and Poor's 500 stock index.

The main purpose of this paper is to compare the performance of three jump-diffusion models whose jump-amplitudes are the log-normally, log-uniformly and log-double-exponentially distributed.

II. Some Theoretical Results

A. Stock Return Process, $\mathbf{S(t)}$

The following stochastic differential equation (SDE) is used to model the dynamics of the asset price, $S(t)$:

$$dS(t) = S(t)\left(\mu_d dt + \sigma_d dW(t) + J(Q)dP(t)\right), \qquad \text{(II.1)}$$

where μ_d is the drift coefficient, σ_d is the diffusive volatility, $W(t)$ is the stochastic diffusion process, $J(Q)$ is the Poisson jump-amplitude, Q is its underlying Poisson amplitude mark process, $J(Q)dP(t)$ is just a symbol for the counting $\sum_{i=1}^{dP(t)} J(Q_i)$, $P(t)$ is the standard Poisson jump process with joint mean and variance $\mathrm{E}[P(t)] = \lambda t = \mathrm{Var}[P(t)]$.

B. Stock Log-Return Process, $\mathbf{ln(S(t))}$

The stock log-return $\ln(S(t))$ can be transformed to a simpler jump-diffusion stochastic differential equation (SDE) upon the use of the stochastic chain rule [6],

$$d[\ln(S(t))] = \mu_{ld}dt + \sigma_d dW(t) + QdP(t), \qquad \text{(II.2)}$$

where $\mu_{ld} \equiv \mu_d - 0.5\sigma_d^2$ is the log-diffusive (ld) drift and for simplicity the log-jump-amplitude is taken as the mark $Q = \ln(J(Q)+1)$.

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C. Log-Normal Jump Distribution

Let the density of the jump-amplitude mark Q be normal

$$\phi_Q(q) = \phi_n(q; \mu_j, \sigma_j^2), \tag{II.3}$$

where $\phi_n(q; \mu_j, \sigma_j^2)$ is the normal density with mean μ_j and variance σ_j^2. The log-normal jump-amplitude jump-diffusion model was used in [9], [1], [3] and others.

For the density for this jump-diffusion model with log-normal jump-amplitude, Hanson and Westman [3] proved, with corrections given here in terms of the distribution truncated to the second order approximation in terms of the Poisson distribution, the following corollary:

Corollary: The second order approximation to $[x_1, x_2]$ bin probability distribution for the linear jump-diffusion log-return increment $\Delta[\ln(S(t))]$ with log-normal jump-amplitude is given by

$$\Phi_{\text{njd}}(x_1, x_2) \simeq \frac{\sum_{k=0}^{2} p_k(\lambda\Delta t)\Phi_n(x_1, x_2; \mu+k\mu_j, \sigma^2+k\sigma_j^2)}{\sum_{j=0}^{2} p_j(\lambda\Delta t)}, \tag{II.4}$$

for $-\infty < x < +\infty$, where $\Phi_n(x_1, x_2; \mu, \sigma^2)$ is the normal distribution on the $[x_1, x_2]$, $\mu \equiv \mu_{ld}\Delta t$, $\sigma \equiv \sqrt{\sigma_d^2 \Delta t}$, $p_k(\Lambda) = e^{-\Lambda}\Lambda^k/k!$ is the Poisson distribution with parameter $\Lambda = \lambda\Delta t$ and k jumps and Δt is the corresponding trading time increment in years.

This corollary is based upon the law of total probability [6, Chapter 0] resulting in the sum over all k Poisson jumps, the convolution theorem [6] yielding the density of the log-jump-diffusion conditioned on there being k IID jumps, and the fact that the convolution of two normals is also normal [6]. For the purpose of comparison, we use two terms of the expansion to provide more accurate estimations since we are dealing with small but moderately small time steps. The additional contribution of the third order approximation is only 1.5% whereas the 2nd order approximation contributed 23% to the 1st. Note that an appropriate 2nd order renormalization is used to preserve the distribution property.

1) Basic Moments $M_i^{(njd)}$ of Log-Return Increments $\mathbf{\Delta[\ln(S(t))]}$ for Log-Normal Jumps for $i = 1{:}4$:

$$\begin{aligned}
M_1^{(\text{njd})} &\equiv \text{E}[\Delta[\ln(S(t))]] = (\mu_{ld}+\lambda\mu_j)\Delta t. \\
M_2^{(\text{njd})} &\equiv \text{Var}[\Delta[\ln(S(t))]] = (\sigma_d^2+\lambda(\sigma_j^2+\mu_j^2))\Delta t. \\
M_3^{(\text{njd})} &\equiv \text{E}\left[(\Delta[\ln(S(t))]-M_1^{(jd)})^3\right] = (3\sigma_j^2+\mu_j^2)\mu_j\lambda\Delta t. \\
M_4^{(\text{njd})} &\equiv \text{E}\left[(\Delta[\ln(S(t))]-M_1^{(jd)})^4\right] \\
&= (\mu_j^4+3\sigma_j^4+6\mu_j^2\sigma_j^2)\lambda\Delta t + 3(\sigma_d^2+\lambda(\sigma_j^2+\mu_j^2))^2(\Delta t)^2.
\end{aligned}$$

D. Log-Uniform Jump Distribution

Let the density of the jump-amplitude mark Q be uniform

$$\phi_Q(q) = \frac{H(b-q) - H(a-q)}{b-a}, \tag{II.5}$$

where $a < 0 < b$ and $H(x)$ is the Heaviside unit step function. The mark Q has moments, $\mu_j \equiv \text{E}[Q] = 0.5(b+a)$, $\sigma_j^2 \equiv \text{Var}[Q] = (b-a)^2/12$. The original jump-amplitude J has mean $\text{E}[J(Q)] = (\exp(b) - \exp(a))/(b-a) - 1$.

For the distribution of the jump-diffusion model with log-uniform jump-amplitude, the following corollary follows from the density in [6, Chapter 5],

Corollary: The second order approximation to $[x_1, x_2]$ bin probability distribution for the linear jump-diffusion log-return increment $\Delta[\ln(S(t))]$ with log-uniform jump-amplitude is given by

$$\Phi_{\text{ujd}}(x_1, x_2) \simeq \frac{\sum_{k=0}^{2} p_k(\lambda\Delta t)\Phi_{\text{ujd}}^{(k)}(x_1, x_2)}{\sum_{j=0}^{2} p_j(\lambda\Delta t)}, \tag{II.6}$$

for $-\infty < x < +\infty$, where $\Phi_{\text{ujd}}^{(0)}(x_1, x_2) \equiv \Phi_n(x_1, x_2; \mu, \sigma^2)$,

$$\begin{aligned}
\Phi_{\text{ujd}}^{(1)}(x_1, x_2) = \frac{1}{b-a}\Bigg(&(x_2-x_1)\Phi(x_2-b, x_2-a; \mu, \sigma^2) \\
&-(x_1-b-\mu)\Phi_n(x_1-b, x_2-b; \mu, \sigma^2) \\
&+(x_1-a-\mu)\Phi_n(x_1-a, x_2-a; \mu, \sigma^2) \\
&+\frac{\sigma}{\sqrt{2\pi}}\left(e^{-\frac{x_{1b}^2}{2}} - e^{-\frac{x_{2b}^2}{2}} - e^{-\frac{x_{1a}^2}{2}} + e^{-\frac{x_{2a}^2}{2}}\right)\Bigg),
\end{aligned}$$

where $x_{1a} = (x_1 - a - \mu)/\sigma$, $x_{2a} = (x_2 - a - \mu)/\sigma$, $x_{1b} = (x_1 - b - \mu)/\sigma$, $x_{2b} = (x_2 - b - \mu)/\sigma$,

$$\begin{aligned}
\Phi_{\text{ujd}}^{(2)}(x_1, x_2) = \frac{0.5\sigma^2}{(b-a)^2}\Bigg(&(x_{2A}^2 - x_{1A}^2)\Phi_n(x_2-C, x_2-A; \mu, \sigma^2) \\
&+(x_{1B}^2 - x_{2B}^2)\Phi_n(x_2-B, x_2-C; \mu, \sigma^2) \\
&+(x_{1A}^2+1)\Phi_n(x_1-A, x_2-A; \mu, \sigma^2) \\
&+(x_{1B}^2+1)\Phi_n(x_1-B, x_2-B; \mu, \sigma^2) \\
&-2(x_{1C}^2+1)\Phi_n(x_1-C, x_2-C; \mu, \sigma^2) \\
&+\frac{1}{\sqrt{2\pi}}(2X_{1C,2C} - X_{1A,2A} - X_{1B,2B})\Bigg),
\end{aligned}$$

where $A = 2a$, $C = a+b$, $B = 2b$, $x_{1A} = (x_1-A-\mu)/\sigma$, $x_{2A} = (x_2-A-\mu)/\sigma$, $x_{1C} = (x_1-C-\mu)/\sigma$, $x_{2C} = (x_2-C-\mu)/\sigma$, $x_{1B} = (x_1-B-\mu)/\sigma$, $x_{2B} = (x_2-B-\mu)/\sigma$, $X_{1A,2A} = x_{1A}e^{-x_{1A}^2/2} - x_{2A}e^{-x_{2A}^2/2}$, $X_{1B,2B} = x_{1B}e^{-x_{1B}^2/2} - x_{2B}e^{-x_{2B}^2/2}$, $X_{1C,2C} = x_{1C}e^{-x_{1C}^2/2} - x_{2C}e^{-x_{2C}^2/2}$.

For the distribution (II.6), the first and second order term formulas, $\Phi_{\text{ujd}}^{(i)}(x_1, x_2)$ for $i = 1{:}2$, have been reduced by integration by parts to single normal distribution integrals to minimize the computational costs, making it comparable for the normal jump case in (II.4). Also, the terms are arranged to minimize the effects of catastrophic cancellation.

1) Fourth Central Moment of Log-Return Increments $\mathbf{\Delta[\ln(S(t))]}$ for Log-Uniform Jumps:

$$\begin{aligned}
M_4^{(\text{ujd})} &= \left(\mu_j^4+1.8\sigma_j^4+6\mu_j^2\sigma_j^2\right)\lambda\Delta t \\
&\quad +3\left(\sigma_d^2+\lambda(\sigma_j^2+\mu_j^2)\right)^2(\Delta t)^2.
\end{aligned}$$

Note that the formulas for the first three moments are the same for both log-normal and log-uniform jumps.

E. Log-Double-Exponential Jump Distribution

Let the density of the jump-amplitude mark Q be double-exponential

$$\phi_Q(q) = \frac{p_1}{\mu_1}\exp\left(\frac{q}{\mu_1}\right)I_{\{q<0\}} + \frac{p_2}{\mu_2}\exp\left(\frac{-q}{\mu_2}\right)I_{\{q\geq 0\}}, \tag{II.7}$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are one-sided means, and $0 < p_1 < 1$ represents the probability of downward jumps while $p_2 = 1 - p_1$ is the probability of upward jumps. The set indicator function is $I_{\{\mathcal{S}\}}$ for set $\mathcal{S}$. The mark Q has moments,

$$\begin{aligned}\mu_j &= \mathrm{E}_Q[Q]=-p_1\mu_1+p_2\mu_2,\\ \sigma_j^2 &= \mathrm{Var}_Q[Q]=p_1\left((\mu_j+\mu_1)^2+\mu_1^2\right)+p_2\left((\mu_j-\mu_2)^2+\mu_2^2\right).\end{aligned}$$

Similar to Corollary II.6, we get the following corollary:
Corollary: The second order approximation to $[x_1,x_2]$ bin probability distribution for the linear jump-diffusion, log-return increment $\Delta[\ln(S(t))]$ with log-double-exponential jump-amplitude is given by

$$\Phi_{\text{dejd}}(x_1,x_2) \simeq \frac{\sum_{k=0}^{2} p_k(\lambda\Delta t)\Phi^{(k)}_{\text{dejd}}(x_1,x_2)}{\sum_{j=0}^{2} p_j(\lambda\Delta t)}, \qquad \text{(II.8)}$$

for $-\infty < x < +\infty$, where

$$\begin{aligned}\Phi^{(0)}_{\text{dejd}}(x_1,x_2) &\equiv \Phi_n(x_1,x_2;\mu,\sigma^2),\\ \Phi^{(1)}_{\text{dejd}}(x_1,x_2) &= \Phi_n(x_1,x_2;\mu,\sigma^2)+p_1(\rho_{x_2,\nu_1}-\rho_{x_1,\nu_1})\\ &\quad +p_2(\rho_{x_1,\nu_2}-\rho_{x_2,\nu_2}),\end{aligned}$$

$$\begin{aligned}\nu_1 &= \mu-0.5\sigma^2/\mu_1,\quad \nu_2=\mu+0.5\sigma^2/\mu_2,\\ \rho_{x_2,\nu_1} &= e^{+(x_2-\nu_1)/\mu_1}\Phi_n(-x_2;-\mu+\sigma^2/\mu_1,\sigma^2),\\ \rho_{x_1,\nu_1} &= e^{+(x_1-\nu_1)/\mu_1}\Phi_n(-x_1;-\mu+\sigma^2/\mu_1,\sigma^2),\\ \rho_{x_1,\nu_2} &= e^{-(x_1-\nu_2)/\mu_2}\Phi_n(x_1;\mu+\sigma^2/\mu_2,\sigma^2),\\ \rho_{x_2,\nu_2} &= e^{-(x_2-\nu_2)/\mu_2}\Phi_n(x_2;\mu+\sigma^2/\mu_2,\sigma^2),\end{aligned}$$

$$\begin{aligned}\Phi^{(2)}_{\text{dejd}}(x_1,x_2) &= \Phi_n(x_1,x_2;\mu,\sigma^2)\\ &\quad +\mu_1\left(\left(p_{12}+p_{11}\left(\mu-\frac{\sigma^2}{\mu_1}+\mu_1-x_2\right)\right)\rho_{x_2,\nu_1}\right.\\ &\quad \left.-\left(p_{12}+p_{11}\left(\mu-\frac{\sigma^2}{\mu_1}+\mu_1-x_1\right)\right)\rho_{x_1,\nu_1}\right)\\ &\quad +\mu_2\left(\left(p_{12}-p_{22}\left(\mu+\frac{\sigma^2}{\mu_2}-\mu_2-x_1\right)\right)\rho_{x_1,\nu_2}\right.\\ &\quad \left.-\left(p_{12}-p_{22}\left(\mu+\frac{\sigma^2}{\mu_2}-\mu_2-x_2\right)\right)\rho_{x_2,\nu_2}\right)\\ &\quad +\frac{\sigma}{\sqrt{2\pi}}(\mu_2p_{22}-\mu_1p_{11})\left(e^{-z_1^2/2}-e^{-z_2^2/2}\right),\end{aligned}$$

$p_{11}=(p_1/\mu_1)^2$, $p_{22}=(p_2/\mu_2)^2$, $p_{12}=2p_1p_2/(\mu_1+\mu_2)$, $z_1=(x_1-\mu)/\sigma$, $z_2=(x_2-\mu)/\sigma$.

1) Third and Fourth Moments of Log-Return Increments $\mathbf{\Delta[\ln(S(t))]}$ for Log-Double-Exponential Jumps:

$$\begin{aligned}M_3^{(\text{dejd})} &= 6(p_2\mu_2^3-p_1\mu_1^3)\lambda\Delta t;\\ M_4^{(\text{dejd})} &= 24(p_2\mu_2^4+p_1\mu_1^4)\lambda\Delta t+3(\sigma_d^2+\lambda(\sigma_j^2+\mu_j^2))^2(\Delta t)^2.\end{aligned}$$

The first and second moments are the same for all three models.

F. Skewness and Kurtosis

Negative skewness and leptokurtosis are considered to be general properties of financial market distributions. Therefore, $M_3^{(\text{jd})}$ and $M_4^{(\text{jd})}$ are needed to get the theoretical skewness and kurtosis coefficient for these three models to sufficient accuracy for a satisfactory comparison.

- Skewness coefficient: $\beta_3^{(\text{jd})} \equiv M_3^{(\text{jd})}/\left(M_2^{(\text{jd})}\right)^{1.5}$.
- Kurtosis coefficient: $\beta_4^{(\text{jd})} \equiv M_4^{(\text{jd})}/\left(M_2^{(\text{jd})}\right)^{2}$.

Sometimes, the kurtosis is represented as the excess kurtosis coefficient by subtracting three from the kurtosis coefficient so that the excess is zero for the normal distribution.

III. Parameter Estimations

The basic point of view, here, is that the financial markets are considered to be a moderate size simulation of one of these three jump-diffusion processes.

A. Empirical Data

We use Standard and Poor's 500 (S&P500) stock index in the decade 1992-2001 [12] as the sample of the financial market since it is in general viewed as one big mutual fund so that it is less dependent on the peculiar behavior of any one stock. Let $n^{(\text{sp})}=2522$ be the number of daily closings $S_s^{(\text{sp})}$ for $s=1{:}n^{(\text{sp})}$, such that there are $ns=2521$ log-returns,

$$\Delta\left[\ln\left(S_s^{(\text{sp})}\right)\right] \equiv \ln\left(S_{s+1}^{(\text{sp})}\right)-\ln\left(S_s^{(\text{sp})}\right) \qquad \text{(III.1)}$$

with empirical average values:

- Mean: $M_1^{(\text{sp})} \simeq 4.015\text{e-}4$.
- Variance: $M_2^{(\text{sp})} \simeq 9.874\text{e-}5$.
- Skewness coefficient: $\beta_3^{(\text{sp})} \simeq -0.2913 < 0$,where $\beta_3^{(n)}=0$ is the normal distribution value and $M_3^{(\text{sp})}$ is the 3rd central log-return moment of the data.
- Kurtosis coefficient: $\beta_4^{(\text{sp})} \simeq 7.804 > 3$,where $\beta_4^{(n)}=3$ is the normal distribution value and $M_4^{(\text{sp})}$ is the 4th central log-return moment of the data.

B. Multinomial Maximum Likelihood Estimation

In a previous paper [7], the multinomial maximum likelihood estimation of model parameters is justified for binned financial data, but applies to very general binned data. The main idea for this method is the following:

- Step 1: Sort the sample data into nb bins and get the sample frequency $f_b^{(sp)}$, for $b=1{:}nb$.
- Step 2: Get the theoretical jump-diffusion frequency with parameter vector $\boldsymbol{x}$:

  $$f_b^{(\text{jd})}(\boldsymbol{x}) \equiv ns\int_{B_b}\phi^{(\text{jd})}(\eta;\boldsymbol{x})d\eta\,,$$

  where $\phi^{(\text{jd})}(\eta;\boldsymbol{x})$ is some jump-diffusion density in η and B_b is the bth bin.
- Step 3: Minimize the objective function:

  $$y(\boldsymbol{x}) \equiv -\sum_{b=1}^{nb}\left[f_b^{(\text{sp})}\ln\left(f_b^{(\text{jd})}(\boldsymbol{x})\right)\right]\,, \qquad \text{(III.2)}$$

  where the negative of the likelihood is minimized, corresponding to the minimizing MATLAB function `fminsearch` to get the optimal parameters $\boldsymbol{x}^*$ for the three compared models, respectively. This MATLAB function is an implementation of the Nelder-Mead down-hill simplex direct search method [11]. The Nelder-Mead is usually faster than other optimization methods when it works. Some comparisons with our multidimensional golden section search method for the financial parameter estimation problem are given in [7].

C. Jump-Diffusion Moment Estimation Constraints

For the jump-diffusion model with log-normal and log-uniform jump-amplitude, there are five (5) free jump-diffusion parameters: $\{\mu_{ld}, \sigma_d^2, \mu_j, \sigma_j^2, \lambda\}$. For the model with log-double-exponential jump-amplitude, there are six (6) free jump-diffusion parameters: $\{\mu_{ld}, \sigma_d^2, \mu_1, \mu_2, p_1, \lambda\}$. So, to reduce this set to a reasonable number, the multinomial maximum likelihood estimation is subjected to the mean and variance constraints:

$$M_1^{(\mathrm{sp})} = M_1^{(\mathrm{jd})} \quad \text{and} \quad M_2^{(\mathrm{sp})} = M_2^{(\mathrm{jd})}\,. \tag{III.3}$$

Two diffusion parameters, μ_{ld} and σ_d, are eliminated by

$$\mu_{ld} = \left(M_1^{(sp)} - \mu_j \lambda \Delta t\right)/\Delta t, \tag{III.4}$$

$$\sigma_d^2 = \max\left[\left(M_2^{(sp)} - \left(\sigma_j^2 + \mu_j^2\right)\lambda\Delta t\right)/\Delta t, \varepsilon\right], \tag{III.5}$$

subject to positivity constraints with sufficiently small $\varepsilon > 0$. Although σ_d^2 normally should be positive, but not necessarily for the first argument of the $\max$ in (III.5). For the log-normal and log-uniform jump-diffusion model, only three free parameters are left: $\boldsymbol{x} = \{\mu_j, \sigma_j^2, \lambda\}$. For the log-double-exponential jump-diffusion model, four free parameters are left: $\boldsymbol{x} = \{\mu_1, \mu_2, p_1, \lambda\}$with significantly more computational estimation costs and it is subject to an exponential form of catastrophic cancellation unless the one-sided exponentials are appropriately collected.

IV. Numerical Results, Figures and Discussion

We use the MATLAB 7.0 [10] to program our codes. The multinomial maximum likelihood estimation given here is used to estimate the jump-diffusion parameters. The numerical optimization was performed using the `fminsearch` function [10]. For the normal distribution integrals $\Phi_n(x_1, x_2; \mu, \sigma^2)$ or $\Phi_n(x_1; \mu, \sigma^2)$, the use of the fast and reliable MATLAB complementary error function `erfc` was critical, since more standard integration functions for unscaled arguments failed small variance σ^2 tests due to poor detection of the main probability mass. For the log-normal and log-uniform model, the same starting point $\boldsymbol{x}_0$ is used. For the log-double-exponential model, the different starting point $\boldsymbol{x}_0$ is used: initial μ_1 and μ_2 are from an initial estimation of the μ_j and using $p_1 \simeq 0.6 > 0.5$ and the $\lambda\Delta t$ value are the same as the log-normal or log-uniform.

The empirical data used in the estimation are the S&P500 daily closing log-returns from the decade 1992-2001. This data is displayed here in Figure 1 with 100 equally spaced bins. The ragged appearance of the histogram resembles the random simulation of a density using a moderate, but inadequate sample size. The crashes are represented by the extreme negative tails and the rallies by the extreme positive tails, but these extreme events are difficult to see since they are rare events with small frequency counts, i.e., the extreme tails are sparsely populated, with the extreme negative events more widely separated than the extreme positive events. Note that extremeness is measured by the values of the log-returns on the horizontal axes.

However, if the histogram frequencies are multiplied by the centered value of the evenly spaced bin log-returns,

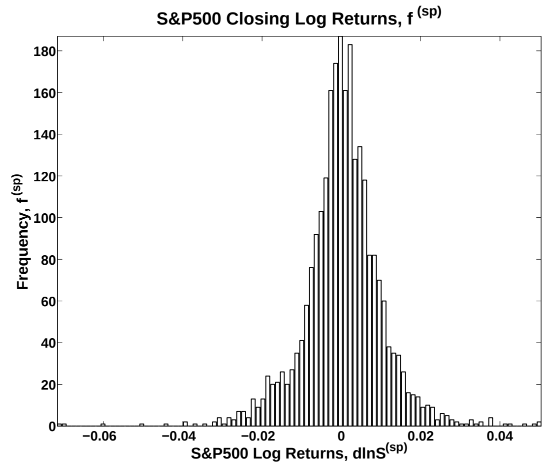

Fig. 1. Histogram of S&P500 log-return frequencies for the decade 1992-2001, using 100 centered evenly spaced bins.

then the extreme jumps are clearly visible. This moment-histogram is called a *hysteriagram* [4] since it magnifies the count of the larger jumps and corresponds to the extreme reaction of investors. The hysteriagram for the S&P500 is given in Figure 2 using the same data as in the previous figure, using the same 100 evenly spaced bins.

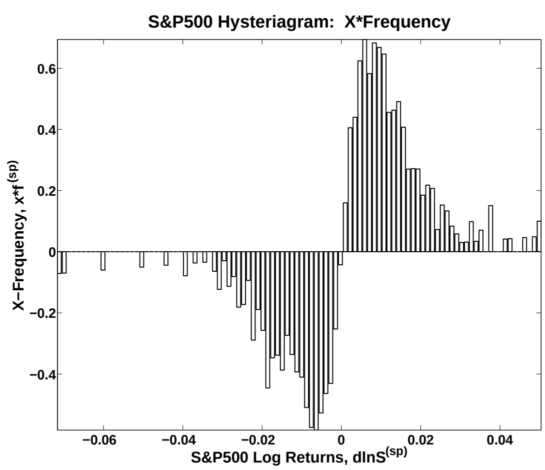

Fig. 2. Hysteriagram of S&P500 log-return frequencies multiplied by the centered bin log-return values for the decade 1992-2001, using 100 evenly spaced bins.

Figure 3 shows that the log-normal jump-amplitude fitted model hysteriagram exhibits too thin tails that decay too fast with the jump magnitude. From (II.4) the bin distribution will be a Poisson sum of normal distributions, so will have the thin Gaussian exponential tails.

Figure 4 shows that the log-uniform jump-amplitude fitted model hysteriagram exhibits thicker tails that decay more slowly with the jump magnitude, especially in the shoulders of the hysteriagram. The convolution of the diffusion with multiple jump uniform distributions in (II.6) help counter the normal distribution tendency to having exponential thin

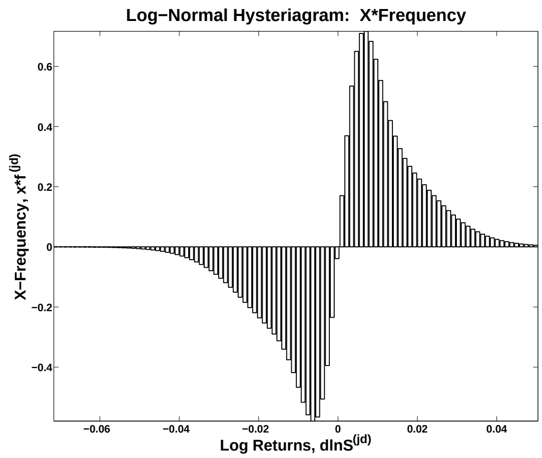

Fig. 3. Hysteriagram of the predicted log-returns frequencies multiplied by the centered bin log-return values for the log-normal jump-amplitude jump-diffusion model, using 100 evenly spaced bins and the estimated parameters.

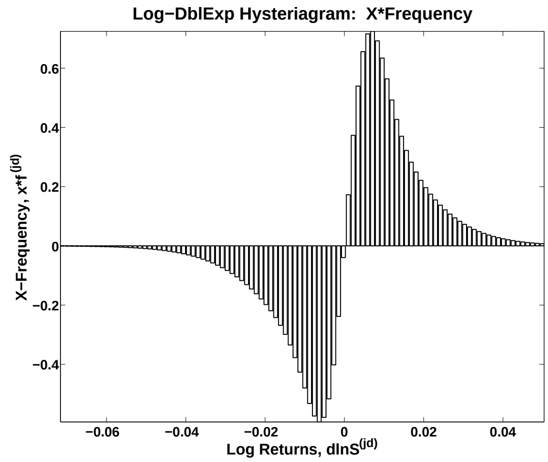

Fig. 5. Hysteriagram of the predicted log-returns frequencies multiplied by the centered bin log-return values for the log-double-exponential jump-amplitude jump-diffusion model, using 100 evenly spaced bins and the estimated parameters.

tails, but not for very large values of the log-returns. The positive tails are thicker since the extreme positive jumps are more closely spaced than the negative jumps. The uniform model mainly compensates for the non-normal data with significant lumps in the shoulders of the jump-diffusion distribution.

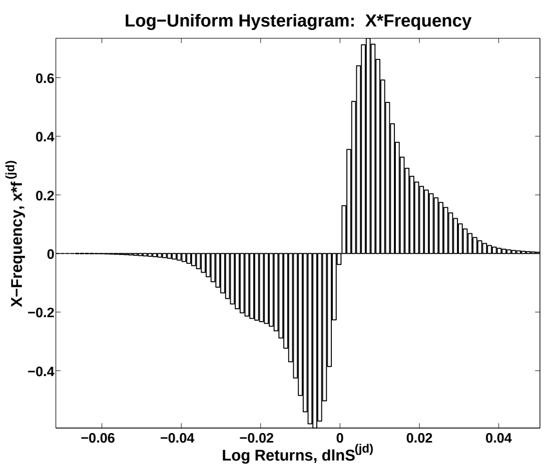

Fig. 4. Hysteriagram of the predicted log-returns frequencies multiplied by the centered bin log-return values for the log-uniform jump-amplitude jump-diffusion model, using 100 evenly spaced bins and the estimated parameters.

Figure 5 shows that the log-double-exponential jump-amplitude fitted model hysteriagram exhibits too thin tails that decay too fast with the jump magnitude that is very similar to the log-normal jump-amplitude model. The convolution of normal and exponential distributions in (II.8), like the normal jump-amplitude model, can only lead to exponential thin tails.

From Table I, we can have a quantitative estimate of the derived distribution parameters $\{\mu_d,\ \sigma_d,\ \mu_j,\ \sigma_j,\ \lambda\}$. Since the trading days per year are about 250 days, it is not likely that the jumps rate is more than 100 per year because the finance market should be kept stable, so, $\lambda \simeq 64$ for the log-uniform is more reasonable. The near-zero peaks of the normal and double-exponential lead to jump rates that are double or more than the uniform jump rate. Note that the jump rate includes all size jumps, including those hidden under the log-normal diffusion. In the table the overall jump mean μ_j is given for the purpose of comparison, but for the double-exponential, the negative jump mean is $-\mu_1 =$ -6.88e-3 and the positive jump mean is $\mu_2 =$ +6.35e-3. For the double-exponential, the probability of negative jumps is $p_1 = 0.504$ and that for positive jumps is $p_2 = 0.496$. The overall jump-diffusion parameters $\{\mu_d, \sigma_d, \mu_j, \sigma_j, \lambda\}$ have somewhat different distributions among the three jump models, with the diffusive means and volatilities being the closest among the parameters. The uniform distribution gives more weight to the negative jumps with the largest $-\mu_j =$ 12.18e-4 and largest $\sigma_j \simeq$ 1.52e-2.

TABLE I

Comparison summary of derived distribution parameters for the log-normal, log-uniform and log-double-exponential jump-diffusion models, respectively.

Model	μ_d	σ_d	μ_j	σ_j	λ
Normal	0.191	0.088	-7.09e-4	1.19e-2	121.
Uniform	0.184	0.100	-12.18e-4	1.52e-2	64.0
Dbl-Exp	0.170	0.085	-3.21e-4	0.94e-2	202.

Table II shows the differences of the variance-normalized higher moments of skewness β_3 and kurtosis β_4 between the estimate value and the observed values. The absolute skewness difference is the lowest for the log-uniform jump model, while the absolute kurtosis difference is the lowest for the log-double-exponential jump model.

On the other hand, the skewness difference is the worst for the log-normal jump model and the kurtosis difference is the worst for the log-uniform jump model, but the log-normal is only 5% lower than the log-uniform. Of course, the numerical calculations of the third and fourth moments are of doubtful computational reliability for data. The final multinomial maximum likelihood values using the negative of minimum values are essentially the same for all models, since the same stopping criterion was used.

TABLE II

The skewness and kurtosis coefficients for the three models are compared to S&P500 values, respectively, and Multinomial Maximum Likelihood (MML $\simeq -\min[y(\mathbf{x})]$).

Model	β_3	%diff.	β_4	%diff.	MML
Normal	-0.147	-49.5	5.98	-23.4	1107.
Uniform	-0.219	-24.7	5.57	-28.7	1106.
Dbl-Exp	-0.183	-37.6	6.80	-12.8	1108.
S&P500	-0.291	0.0	7.80	0.0	—

Table III shows that the log-normal model takes somewhat more iterations and function evaluations than the log-uniform model does, but the log-normal model parameter estimate takes about 60% of the time to execute. One reason is that the log-normal requires only similar normal distribution calculations for each jump count k in (II.4), while the others require more complex combinations of normal distributions, powers and exponentials. The extra parameter needed for the double-exponential means the iteration count, the function evaluation count and the timings are much greater. However, the computational effort for both the uniform and double-exponential models have been greatly reduced by using integration by parts and more to get single integrals. If the parameter estimation is done off-line for an application instead of on-line then the computational saving would not be too much of an advantage.

TABLE III

Comparison summary of computational performance measures:

Jump Model	Number Parms.	Number Iters.	Function Evals.	Timings (sec)
Normal	3	131	238	4.7
Uniform	3	71	128	7.8
Dbl-Exp	4	205	343	21.

Combined Legend for Table I, Table II and Table III:

- Maximum Number of Iterations: 400.
- Using same fminsearch tolerances: tolx = 5e-7 and toly = 5e-7.
- Using a dual G5@2GHz CPU computer processor with MATLAB 7.0.

V. Summary and Conclusions

From the above theoretical and data analysis, we can get the following conclusions:

- The log-uniform model is the qualitatively best overall among the three models, in terms of genuinely representing the fat tail property, better approximation to the empirical skewness and more reasonable jump rates of real-world market distributions.
- The log-normal model runs faster than the other two models due to simpler normal bin integrals. On the other hand, the integration by parts technique has been used to reduce the computational effort for the log-uniform and log-double-exponential models. However, the deficiencies of the log-normal model demonstrates that the distribution that is better analytically is not necessarily a better model for financial markets.
- The results for the log-normal and log-double-exponential jump amplitude models are qualitatively similar, having exponentially small tails and near-zero peaks in the jump distribution making small jumps more likely.
- However, all three models give reasonable quantitative, although somewhat mixed, results.
- For the future research and considerations.
  1) To improve the log-uniform and other jump models, the stochastic volatility will be considered with other factors.
  2) To consider the option price and optimal portfolio applications, approximate solutions to these problems will be obtained.

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